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Invitied Editorial

The Impact of COVID on Accessing Critical Care Services

The COVID- 19 pandemic has spread rapidly around the world and resulted in significant unexpected morbidity and mortality. It has created an acute public health emergency of greater magnitude across the globe. The rapid spread of the pandemic has posted challenges to both high and the low-income countries and even shattered the world's top economies. The negative emotions of despair, fear and anxiety have resulted from the considerable uncertainty of new virus that has unleashed into the global society, especially in the absence of effective evidence-based treatment or a freely available vaccine to prevent it. The main modality that has adopted by most health authorities across the world to manage and control the spread of the virus has been the stringent public health measures, including social distancing, lock downs and travel restrictions. It has been learnt that there are significant drawbacks and challenges in the horizon of finding new treatment options and making vaccines freely available. Until the availability of such therapeutics or vaccines, it is the responsibility of the societies to learn and adapt to manage the uncertainties. However, it is inevitable that the consequences of such preventive measures would have great indirect impact on many aspects of day to day life. Some of them even would be endangering to the mankind. Access to critical health care services is a basic need of a human to safeguard the life. Several reports around the globe during the period of lockdown have highlighted the fact that access to such healthcare services have been hindered and resulted in significant morbidity and the mortality which was much worse than that of the direct effect of the pandemic prevailed in that region. Care of non-communicable diseases such as diabetes, hypertension, heart disease, stroke, asthma, epilepsy and cancer by limiting their access to preventive and therapeutic services during the restrictive period were significantly affected. The impact of lockdowns could also affect the health of populations in the long term due to loss of livelihoods, reduced mental wellbeing and a rebound post lockdown effect on many such non-communicable diseases.

Although Sri Lanka has been classified under the category of an Upper-Middle Income country, it has a highly devolved public health system with higher ratings in many health care indices comparable to developed nations. The country has shown promising results in controlling the pandemic to date. However, the local statistics has pointed out that the indirect impact on non COVID-19 diseases could turn out to be greater than the health effects of COVID-19, especially in disadvantaged and vulnerable populations in the community.

Studies from Northern Sri Lanka during the lockdown period have shown significant reduction in accessing healthcare services as evidence by drop in admissions to Medical wards of both in the tertiary care centre like Teaching Hospital Jaffna and the peripherally located primary health institutions in the peninsula. Significant reductions noted in the number of coronary angiograms performed and cardiac interventions done in the coronary care units. A similar drop was noted in the number of new referrals for cancer care and patients treated with chemotherapy and radiotherapy at Tellipalai Trail Cancer Hospital(TTCH). This dramatic drop in the utilization of healthcare accompanied by a reduction of hospital deaths by more than 60% indicates that during this restrictive period many vulnerable and disadvantage population could have succumbed to non COVID diseases at home and the effects of defaulting medication adherence and follow up could have a huge impact in the public health system.

Reasons attributed the decreased access to critical care by the population are multifactorial. Lack of public transport and restrictions in travel between the various regions during restrictive period may have led to the difficulties in accessing cardiac and cancer care. The special task force employed to control the spread played a critical role in all the anti COV measures such as the control of airports, running of the quarantine camps and the rigorous enforcement, including prison sentences, for any non-compliance with the law as a result Sri Lanka did not face a surge of COV patients. Allocation of significant resources to COVID control including man power also had a great impact in providing essential services to the needy patients. In addition, the spread of information on the added risk of chronic illness and cancer patients in the outcome of COVID had also created fear and anxiety in the society and the lack of knowledge on the importance of maintaining continuity of care in these disorders resulted in the reluctance of patients to seek routine medical care. An extra ordinary importance given on social media describing COVID as the “deadliest of all prevailing diseases” not only created fear in the community but also introduced panic among patients with a febrile illness to approach health care as they would be suspected to have COVID.

The experience from Sri Lanka indicates that decisions around enforcing and easing restrictive measures need to be specific for different countries and regions, based on the socio-economic status of the population, availability of access to healthcare services and extent of the direct impact of COVID. Targeted and balanced dissemination of information are needed, especially in countries like Sri Lanka to manage the COV pandemic optimally and to encourage the vulnerable populations to health care pathways.

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Leading Article

Second Order Conditions for Kuhn-Tucker Sufficient Optimality for Optimization Problems with Linear Matrix Inequality Constraints

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Abstract:

The Kuhn-Tucker Sufficiency Theorem states that a feasible point that satisfies the Kuhn-Tucker conditions is a global minimizer for a convex programming problem for which a local minimizer is global. In this paper, we present a new second order conditions for Kuhn-Tucker sufficiency for minimizing a smooth function with linear matrix inequality constraints and bounds on the variables. In particular, we provide a necessary and sufficient conditions for a local minimizer to be a global minimizer over a box when the objective function is weighted sum of squares and linear functions. Numerical examples are given to illustrate the significance of sufficiency criteria.

Keywords: Smooth nonlinear programming problems, linear matrix inequality constraints, Kuhn-Tucker conditions, sufficient global optimality.

1 Introduction

In this paper, we develop conditions under which a Kuhn-Tucker point is a global minimizer of a multi-extremal smooth mathematical semidefinite programming model problem:

$$(LIMP) \qquad \min_{x\in\mathbb{R}^n} f(x)$$

$$s.t \qquad F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0$$

$$x_{\,i} \in [u_i,\, v_i];\ i\ = 1,2, \ldots \ldots n,$$

Where $f:\ \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable function on an open set containing a compact set $\Delta = \{(x_1, \ldots, x_n)^T | x_i \in [u_i, v_i] \text{ for } i = 1, \ldots..n\}$, and $u_i \le v_i,\ i = 1, \ldots..n,\ F_i \in S^m,\ \ i = 1, \ldots..m$ and S^m is the set of all symmetric $m \times m$ matrices. The linear matrix inequality (LMI) constraint, $F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0$ means that the matrix $F_0 + \sum_{i=1}^{n} x_i F_i$ is positive semidefinite. Optimization model problems with (LMI)constraints are also known as semidefinite optimization

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problems $(9, 10)$. Model problems of the form $(LIMP)$ cover large classes of nonconvex continuous optimization problems. Various generalized convexity conditions such as pseudo-convexity and quasi-convexity, just to name a few, have been given in the literature for a Kuhn-Tucker point to be a global minimizer of a nonlinear programming problem $(1, 4, 8, 11)$ and they often apply to problems where a local minimum is global. These Kuhn-Tucker sufficiency criteria have limited value for multi-extremal optimization problems.

The purpose of this paper is to present sufficient conditions for a given Kuhn- Tucker point to be a global minimizer of the general optimization model problem with (LMI) constraints; $(LMIP)$. We obtain sufficient conditions for global optimality by constructing quadratic underestimators then by characterizing global minimizers of the underestimators. As a special case, we apply the results obtained to quadratic programming problems. In particular, we provide a necessary and sufficient conditions for a local minimizer to be a global minimizer over a box when the objective function is weighted sum of squares and linear functions. We discuss examples to illustrate the significance of the optimality conditions, presented in this paper.

2 Sufficient Global Optimality conditions

In this section we obtain sufficient global optimality conditions for smooth nonconvex minimization problem $(LMIP)$ with linear matrix inequality constraints, by the method of underestimation.

We begin by presenting definitions and notations that will be used throughout this section. $F_i \in S^m$, $i = 1, \ldots\ldots m$ and S^m is the space of all symmetric $m \times m$ matrices.

Let $$S_+^m = \{M \in S^m | M \geq 0\}$$

$$\Gamma = \{x \in \mathbb{R}^n | F_0 + \sum_{i=1}^{n} x_i F_i \in S_+^m\}$$

and $$D := \Gamma \cap \Delta \tag{1}$$

Let, $$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i\ ,\quad \hat{F}(x) = \sum_{i=1}^{n} x_i F_i\ , x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Then $\hat{F}(\ )$ is a linear operator from $\mathbb{R}^n$ to S^m and its dual is defined by

$$\hat{F}^*(M) = (tr[F_1 M], \ldots, tr[F_n M])^T \text{for any } M \in S^m,$$

where $tr[\,]$ is the trace operation. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in D$. Define, for each *i=1,2,.....n.*

$$\tilde{x}_i := \begin{cases} -1 & \text{if } \bar{x}_i = u_i \\ 1 & \text{if } \bar{x}_i = v_i \\ (\nabla f(\bar{x}) - \hat{F}^*(M))_i & \text{if } \bar{x}_i \in (u_i, v_i) \end{cases}$$

If $\bar{x} = (\bar{x}_1, \ldots\ldots, \bar{x}_n) \in D$ is a local minimizer of $(LIMP)$ and if a certain constraint qualification holds then the following Kuhn-Tucker conditions hold:

$$(\exists M \in S_+^m \text{ such that } tr[MF(\bar{x})] = 0) \text{ and } (\nabla f(\bar{x}) - \hat{F}^*(M))^T (x - \bar{x}) \geq 0, \forall x \in \Delta. \quad \textbf{(2)}$$

The condition (2) can equivalently be written as

$$(\exists M \in S_+^m \text{ such that } tr[MF(\bar{x})] = 0) \text{ and } \tilde{x}_i(\nabla f(\bar{x}) - \hat{F}^*(M))_i \leq 0, \forall i = 1, \ldots\ldots, n. \quad \textbf{(3)}$$

We now state a second order sufficiency conditions for a Kuhn Tucker point to be a global minimizer of $(LIMP)$.

Theorem 2.1 *For $(LMIP)$; let $\bar{x} \in D = \Gamma \cap \Delta$ and the Kuhn-Tucker conditions hold at $\bar{x}$ with multiplier $M \in S_+^m$. If, for each $x \in D$,*

$$[SC] \qquad \nabla^2 f(x) - diag\left(\frac{2\tilde{X}_1 \nabla\left(f(\bar{x}) - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \ldots, \frac{2\tilde{X}_n\left(\nabla f(x) - \hat{F}^*(M)\right)_n}{v_n - u_n}\right) \geq 0$$

then $\bar{x}$ is a global minimizer of $(LMIP)$.

Proof. Let $Q = diag\left(\frac{2\tilde{X}_1 \nabla\left(f(\bar{x}) - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \ldots, \frac{2\tilde{X}_n\left(\nabla f(x) - \hat{F}^*(M)\right)_n}{v_n - u_n}\right)$

Define a quadratic function $g: \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) := \frac{1}{2} x^T Q_x + (\nabla f(\bar{x}) - Q(\bar{x}) - \hat{F}^*(M))^T x. \quad (4)$$

Let $l(x) := f(x) - \hat{F}^*(M)^T x - tr(MF_0),\ x \in \mathbb{R}^n$ and $\emptyset(x) := l(x) - g(x), x \in \mathbb{R}^n$. Then it is easy to see that

$$\nabla\emptyset(x) = 0 \text{ and } \nabla^2\emptyset(x) = \nabla^2 f(x) - Q \geq 0 \quad \text{for all } x \in \Delta.$$

Therefore, $\emptyset(x) - \emptyset(\bar{x}) \geq 0$ for all $x \in \Delta$, as $\emptyset$ is a convex function.

$$\text{Thus} \qquad l(x) - l(\bar{x}) \geq g(x) - g(\bar{x}), \forall x \in \Delta \quad \textbf{(5)}$$

Since $M \in S^m$ and $F(x) \in S_+^m$ for all $x \in \Gamma$, we have $tr[MF(x)] \geq 0$, for all $x \in D = \Delta \cap \Gamma$.

Hence, for all $x \in D$,

$$\begin{aligned} f(x) - f(\bar{x}) &\geq f(x) - tr[MF(x)] - f(\bar{x}) \\ &= f(x) - tr[MF(x)] - (f(\bar{x}) - tr[MF(\bar{x})]) \\ &= (l(x) - l(\bar{x}) \end{aligned}$$

Therefore from (5), we have

$$f(x) - f(\bar{x}) \geq g(x) - g(\bar{x}) \text{ for all } x \in D. \tag{6}$$

Now,

$$g(x) - g(\bar{x}) = \sum_{i=1}^{n} \left(\frac{\tilde{X}_1 \nabla \left(f(\bar{x}) - \widehat{F}^*(M) \right)_i}{v_i - u_i} \right) (x_i - \bar{x}_i)^2 + \left(\nabla f(\bar{x}) - \widehat{F}^*(M) \right)_i (x_i - \bar{x}_i)$$

We claim that $g(x) - g(\bar{x}) \geq 0, \forall x \in \Delta$ if and only if for each $i = 1, \ldots, n$

$$\sum_{i=1}^{n} \left(\frac{\tilde{X}_1 \nabla \left(f(\bar{x}) - \widehat{F}^*(M) \right)_i}{v_i - u_i} \right) (x_i - \bar{x}_i)^2 + \left(\nabla f(\bar{x}) - \widehat{F}^*(M) \right)_i (x_i - \bar{x}_i) \geq 0 \tag{7}$$

Indeed, if there exists i_0 and x_{i0} such that (7) is not fulfilled, by taking $x = (\bar{x}_1 \ldots, \bar{x}_{i0-1}, \bar{x}_{i0}, \bar{x}_{i0+1}, \ldots, \bar{x}_{n,})$, we have $\bar{x} \in \Delta$ and

$$g(x) - g(\bar{x}) = \sum_{i=1}^{n} \left(\frac{\tilde{X}_1 \nabla \left(f(\bar{x}) - \widehat{F}^*(M) \right)_i}{v_i - u_i} \right) (x_i - \bar{x}_i)^2 + \left(\nabla f(\bar{x}) - \widehat{F}^*(M) \right)_i (x_i - \bar{x}_i) < 0$$

We now show that (7) holds by considering following three cases.

Case 1: $\bar{x}_i = u_i$. Then, $(\nabla f(\bar{x}) - \hat{F}^*(M))_i \geq 0$. Thus,

$$\sum_{i=1}^{n} \left(\frac{\tilde{X}_1 \nabla \left(f(\bar{x}) - \widehat{F}^*(M) \right)_i}{v_i - u_i} \right) (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \widehat{F}^*(M) \right)_i \geq 0, \forall x_i \in [u_i, v_i])$$

Hence (7) holds.

Case 2: $\bar{x}_i = v_i$. Then, $(\nabla f(\bar{x}) - \hat{F}^*(M))_i \leq 0$. Thus,

$$\sum_{i=1}^{n} \left(\frac{\tilde{X}_1 \nabla \left(f(\bar{x}) - \widehat{F}^*(M) \right)_i}{v_i - u_i} \right) (x_i - \bar{x}_i) + \left(\nabla f(\bar{x}) - \widehat{F}^*(M) \right)_i \leq 0, \forall x_i \in [u_i, v_i])$$

Hence (7) holds.

Case 3: $\bar{x}_i \in (ui, vi)$. Then, , $(\nabla f(\bar{x}) - \hat{F}^*(M))_i = 0$ and clearly, (7) holds.

By combining all the above three cases, we have (7) holds and the conclusion follows.

For $x \in D$, denote the eigenvalues of $\nabla^2(x)$ by $\delta i(x), i = 1,2,\dots,n$. Let

$$\delta^* = \min_{x\in D} \min_{i=1,\dots,n} \delta_i(x)$$

In the case where f is a quadratic function with the constant Hessian A, δ^* denotes the least eigen value of A. We also observe that δ^* is often used as a "convexifier" for "convexifing" a twice differentiable function by a quadratic term. In the following lemma we provide sufficient optimality conditions in terms of δ^*.

Corollary 2.1 *For* $(LMIP)$*, let* $\bar{x} \in D = \Gamma \cap \Delta$*. Suppose that there exist* $M \in S_+^m$ *such that* $tr[MF(\bar{x})] = 0$ *and* $\tilde{\chi}_i(\nabla f(\bar{x}) - \hat{F}^*(M))_i \le 0, \forall\, i = 1,\dots,n$*. If*

$$[SC1] \qquad \delta^* + \min_{i=1,\dots,n} \left\{ \frac{2\tilde{X}_\mathrm{i} \left(\nabla f(\bar{x}) - \hat{F}^*(M)\right)_i}{v_i - u_i} \ge 0 \right\}$$

then $\bar{x}$ *is* a global minimizer of $(LMIP)$.

Proof. Let, $\lambda(x)$ be the least eigen value of

$$\nabla^2 f(x) - diag\left(\frac{2\tilde{X}_1 \nabla \left(f(\bar{x}) - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \dots, \frac{2\tilde{X}_n \left(\nabla f(x) - \hat{F}^*(M)\right)_n}{v_n - u_n} \right).$$

By the variational characterization of the least eigen value, we have,

$$\lambda(x) \ge \min_{i=1,\dots,n} \delta_i(x) + \min_{i=1,\dots,n} \left(\frac{-2\tilde{X}_\mathrm{i} \left(\nabla f(\bar{x}) - \hat{F}^*(M)\right)_i}{v_i - u_i} \right).$$

Therefore, for each $x \in D$,

$$\lambda(x) \ge \min_{i=1,\dots,n} \delta^*(x) + \min_{i=1,\dots,n} \left(\frac{-2\tilde{X}_\mathrm{i} \left(\nabla f(\bar{x}) - \hat{F}^*(M)\right)_i}{v_i - u_i} \right) \ge 0$$

Thus, for each $x \in D$,

$$\nabla^2 f(x) - diag\left(\frac{2\tilde{X}_1\nabla\left(f(\bar{x}) - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \dots, \quad \frac{2\tilde{X}_n\left(\nabla f(x) - \hat{F}^*(M)\right)_n}{v_n - u_n}\right) \geq 0$$

Hence, the conclusion follows from Theorem(2.1).

3 Quadratic minimization

In this section, we consider model problems $LMIP$ with quadratic objective function:

$$(LIMP - Q) \quad \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Ax + a^T x$$

$$s.t \qquad F_0 + \sum_{i=1}^{n} x_i F_i \geq 0$$

$$x_i \in [u_i, v_i];\ i = 1,2,\dots\dots.n,$$

where $A = (a_{ij}) \in S^n$ and $a \in \mathbb{R}^n$.

Theorem 3.1 *For $(LMIP - Q)$, let $\bar{x} \in D$. Suppose that $M \in S_+^m$ such that $tr[MF(\bar{x})] = 0$, and $\chi_i(A\bar{x} + a - \hat{F}^*(M))_i \leq 0, \forall\, i = 1,\dots,n$. Let μ be the least eigen value of A. If*

$$\mu + \min_{i=1,\dots,n}\left(\frac{-2\tilde{X}_\mathrm{i}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_i}{v_i - u_i}\right) \geq 0$$

then $\bar{x}$ is a global minimizer of $(LMIP)$.

Proof: By the variational characterization of the least eigen value , we have

$$\min_{v^T v=1} v^T Av + \min_{i=1,\dots,n}\left(\frac{-2\tilde{X}_\mathrm{i}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_i}{v_i - u_i}\right) \geq 0$$

Thus,

$$\min_{v^T v=1} v^T\left(A - \mathrm{diag}\left(\frac{2\tilde{X}_1\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \dots \frac{2\tilde{X}_\mathrm{n}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_n}{v_n - u_n}.\right)\right)v \geq 0$$

Hence the least eigen value of

$$A - \mathrm{diag}\left(\frac{2\tilde{X}_1\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \dots \frac{2\tilde{X}_\mathrm{n}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_n}{v_n - u_n}.\right) \geq 0$$

Thus,

$$A - \operatorname{diag}\left(\frac{2\tilde{X}_1\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_1}{v_1 - u_1}, \dots \frac{2\tilde{X}_\mathrm{n}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_n}{v_n - u_n}.\right) \geq 0$$

Hence, the conclusion follows from Theorem 2.1.

Theorem 3.2 *For* $(LMIP - Q)$*, let* $\bar{x} \in D$*. Suppose that* $M \in S_+^m$ *such that* $tr[MF(\bar{x})] = 0$*, and* $\chi_i(A\bar{x} + a - \hat{F}^*(M))_i \leq 0, \forall\, i = 1,\dots,n$*. If, for each* $i = 1,\dots,n$*,*

$$|a_{ij}| - \sum_{i\neq j:j=1}^{n} |a_{ij}| \geq \frac{-2\tilde{X}_\mathrm{n}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_n}{v_n - u_n} \qquad (\mathbf{8})$$

then $\bar{x}$ *is a global minimizer of* $(LMIP)$*.*

Proof. Since, $\chi_i(A\bar{x} + a - \hat{F}^*(M))_i \leq 0, \forall\, i = 1,\dots,n$, (8) implies that,

$$a_{ii} - \frac{2\tilde{X}_\mathrm{i}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_i}{v_i - u_i} \geq |a_{ii}| - \left|\frac{2\tilde{X}_\mathrm{i}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_i}{v_i - u_i}\right|$$

$$= |a_{ii}| + \frac{2\tilde{X}_\mathrm{i}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_i}{v_i - u_i}$$

$$\geq \textstyle\sum_{i\neq j:j=1}^{n} |a_{ij}|$$

Therefore matrix

$$A - \operatorname{diag}\left(\left(\frac{2\tilde{X}_1\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_1}{v_1 - u_1}\right), \dots \frac{2\tilde{X}_\mathrm{n}\left(\mathrm{A}\bar{x} + a - \hat{F}^*(M)\right)_n}{v_n - u_n}.\right)$$

is diagonally dominant x and hence positive semi-definite. So, the conclusion follows from Theorem(2.1).

We now consider a special case of $(LMIP - Q)$ where the objective function is weighted sum of squares with A is a diagonal matrix, $A = diag(a_{11},\dots,a_{nn})$ and without any constraints, over a box:

$$(QW) \qquad \min_{x\in\mathbb{R}^n} \sum_{i=1}^{n} \frac{1}{2} a_{ii}^2 x_i^2 + a_i x_i$$

$$x_{\,i} \in [u_i,\, v_i];\ i = 1,2,\dots\dots.n,$$

We give a necessary and sufficient conditions for a Kuhn Tucker point $\bar{x}$ to be a global minimizer of (QW).

Theorem 3.3 *Let $\bar{x}$ be a local minimizer of (QW).. Then, $\bar{x}$ is a global minimizer of (QW). if and only if*

$$a_{ii}(v_i - u_i) - 2\tilde{\chi}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall\, i = 1, \ldots, n. \tag{9}$$

Proof. Suppose that

$$a_{ii}(v_i - u_i) - 2\tilde{\chi}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall\, i = 1, \ldots, n.$$

Then,

$$A - \operatorname{diag}\left(\left(\frac{2\tilde{X}_1\,(\mathrm{A}\bar{x} + a)_1}{v_1 - u_1}\right), \ldots \frac{2\tilde{X}_\mathrm{n}\,(\mathrm{A}\bar{x} + a)_n}{v_n - u_n}.\right) \geq 0$$

where $A = diag(a_{11}, \ldots, a_{nn})$. Thus, by applying Theorem 2.1, $\bar{x}$ becomes a global minimizer of (QW). Suppose that $\bar{x}$ is a global minimizer of (QW), then by Theorem 2.1 of (Jeyakumar, *et al.*, 2006), it is necessary that,

$$\frac{1}{2}\max\{0, -a_{ii}\}\,(v_i - u_i) + \tilde{\chi}_i(a_{ii}\bar{x}_i + a_i) \leq 0, \forall i = 1, \ldots \ldots n$$

Hence,

$$\frac{1}{2} - a_{ii}(v_i - u_i) + \tilde{\chi}_i(a_{ii}\bar{x}_i + a_i) \leq 0, \forall i = 1, \ldots \ldots n$$

Therefore, $a_{ii}(v_i - u_i) - 2\tilde{\chi}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall\, i = 1, \ldots, n.$ and the conclusion follows.

We now apply Theorem 3.1 to the following Fractional programming Problem:

$$(FP) \qquad \min_{x \in \mathbb{R}^n} \frac{\sum_{i=1}^{n} \frac{1}{2} a_i^2 x_i^2 + b_i x_i}{\sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i}$$

$$x_{\,i} \in [u_i,\, v_i];\; i = 1, 2, \ldots \ldots . n,$$

where, a_i, b_i, c_i, d_i, u_i, v_i $\in$ $I\!R$ $i = 1, \ldots, n$, and, for each $x \in D$, $\sum_{i=1}^{n} \frac{1}{2} a_i x_i^2 + b_i x_i \geq 0$ and $\sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i \geq 0.$

Theorem 3.4 *Let $\bar{x}$ be a local minimizer of (FP). Then, $\bar{x}$ is a global minimizer of (FP) if and only if*

$$(a_i - q(\bar{x})c_i)(v_i - u_i) - 2\tilde{\chi}_i(a_i\bar{x}_i + b_i - q(\bar{x})(c_i\bar{x}_i + d_i)) \geq 0, \forall\, i = 1, \ldots, n.$$

Proof. For $x \in D$, define

$$q(x) = \frac{\sum_{i=1}^{n} \frac{1}{2} a_i^2 x_i^2 + b_i x_i}{\sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i}$$

Note that, $\bar{x}$ is a global minimizer/local minimizer of (FP) if and only if $\bar{x}$ is a global minimizer/local minimizer of the following minimization of the form (QW):

$$(FPW) \qquad \min_{x\in\mathbb{R}^n}\sum_{i=1}^{n}\frac{1}{2}a_i^2x_i^2+b_ix_i-q(\bar{x})\left(\sum_{i=1}^{n}\frac{1}{2}c_i^2x_i^2+dx_i\right)$$

By applying Theorem 3.1, $\bar{x}$ is a global minimizer of (FPW) if and only if

$$(a_i - q(\bar{x})c_i)(v_i - u_i) - 2\tilde{\chi}_i(a_i\bar{x}_i + b_i - q(\bar{x})(c_i\bar{x}_i + d_i)) \geq 0, \forall\, i = 1,\ldots,n.$$

Hence, the conclusion follows.

Finally, we apply Theorem 2.1 when inequality constraints are replaced by the standard linear inequalities:

$$(LIP) \quad \min_{x\in\mathbb{R}^n} f(x) \quad s.t. \quad b_0+Bx\geq 0$$
$$x_{\,i} \in [u_i,\, v_i];\ i = 1,2,\ldots\ldots.n,$$

where $B = (b_{ij})$ is an $m \times n$ matrix and $b_0 = (b_{01},\ldots,b_{0m})^T.\lambda \in \mathbb{R}_+^m$

Corollary 3.1 *Let* $\bar{x} \in D$. *Suppose that,* $\lambda \in \mathbb{R}_+^m$, *such that* $\lambda^T(b_0+Bx) = 0$ *and* $\chi_i(\nabla f(\bar{x}) - B\lambda)_i \leq 0, \forall\, i = 1,\ldots,n$. *If, for each* $x \in \Delta$,

$$\nabla^2 f(x) - \operatorname{diag}\left(\frac{2\tilde{X}_1\ (\mathrm{A}\bar{x}+a)_1}{v_1-u_1},\ldots,\frac{2\tilde{X}_\mathrm{n}\ (\mathrm{A}\bar{x}+a)_n}{v_n-u_n}.\right)\geq 0$$

then $\bar{x}$ *is a global minimizer of* (LIP).

Proof. For each $i = 0,1,\ldots,n,$ let $F_i = diag(b_{i1},\ldots,b_{im})$. Let $M = diag(\lambda)$. Then $\hat{F}^*(M) = B\lambda$. Then,by applying Theorem 2.1 the conclusion follows.

Example 1 *Consider the following smooth minimization model problem*

$$\underset{x\in\mathbb{R}^2}{\mathrm{Min}}\, f(x) = x_1^2x_2+x_1x_2^2-x_1-x_2 \quad s.t \quad F_0+\sum_{i=1}^{2}x_i\,F_i\geq 0\,,$$

$$x\in\Delta= [-1,0]\times[-1,0]$$

where

$$F_0=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix},\ F_1=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix} \text{ and } F_2=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Let $\bar{x} = (\bar{x}_1,\bar{x}_2) = (0,0) \in \Delta$. *Clearly* $\bar{x} \in D$.

We now check for $\bar{x} = (0,0)$:

Let $z \in \Delta$.

$$Then\ F(\bar{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \nabla f(\bar{x}) = (-1,-1)^T$$

$$And\ \nabla^2 f(z) = \begin{pmatrix} 2z_2 & 2z_1 + 2z_2 \\ 2z_1 + 2z_2 & 2z_1 \end{pmatrix}$$

$$Taking\ M = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, we\ obtain$$

$$M \in S_+^3,\ tr(MF(\bar{x}) = 0,\ \hat{F}^*(M) = (4,2)^T\ and\ \tilde{\chi}_i(\nabla f(\bar{x})\ +\ \hat{F}^*(M))_i \geq 0, \forall\ i\ =\ 1, \ldots, n.$$

$$H_z = \nabla^2 f(z) - diag(2\tilde{\chi}_1(\nabla f(\bar{x})\ +\ \hat{F}^*(M)), \tilde{\chi}_2(\nabla f(\bar{x})\ +\ \hat{F}^*(M))_2$$

$$= \begin{pmatrix} 2z_2 + 10 & 2z_1 + 2z_2 \\ 2z_1 + 2z_2 & 2z_1 + 6 \end{pmatrix}$$

Now $(2z_2 + 10) > 0, \forall (z_1, z_2) \in \Delta$

$$\det \begin{pmatrix} 2z_2 + 10 & 2z_1 + 2z_2 \\ 2z_1 + 2z_2 & 2z_1 + 6 \end{pmatrix} = 60 = 20z_1 + 12z_2 + 4z_1z_2 - 4z_1^2 - 4z_2^2$$

$$> 0, \forall (z_1, z_2) \in \Delta$$

Therefore H_z is positive semidefinite for each $(z_1,z_2) \in \Delta$. Hence we see that $[SC]$ is satisfied at $\bar{x}$ = $(0,0)$ which is a global minimizer of $(E2)$.

Example 2 *Consider the following smooth minimization model problem (QW):*

$$(E2) \qquad \min_{x \in \mathbb{R}^3} f(x) = -x_1^2 + x_2^2 - 3x_3 - x_1 - x_2,$$

$$s.t \quad x \in \Delta = [-1,1] \times [-1,1] \times [-1,1]$$

$(-1,0,-1), (-1,0,1), (1,0,-1), (1,0,1)$ *are the local minimizers of $E(2)$. It is easy to check that, the condition (9) is satisfied only at $\bar{x}\ =\ (1,0,1)$ and $\bar{x}$ is indeed the global minimizer of $E(2)$*

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Research Article

Geo-statistical approach for prediction of groundwater quality in Chunnakam aquifer, Jaffna Peninsula

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Abstract

Chunnakam aquifer is the main limestone aquifer of Jaffna Peninsula. The population of the Jaffna Peninsula depends entirely on groundwater resources to meet all of their water requirements. Thus for protecting groundwater quality in Chunnakam aquifer, data on spatial and temporal distribution are important. Geostatistics methods are one of the most advanced techniques for interpolation of groundwater quality. In this study, Ordinary Kriging and IDW methods were used for predicting spatial distribution of some groundwater characteristics such as: Electrical Conductivity (EC), pH, nitrate as nitrogen, chloride, calcium, carbonate, bicarbonate, sulfate and sodium concentration. Forty four wells were selected to represent the entire Chunnakam aquifer during January, March, April, July and October 2011 to represent wet and dry season within a year.After normalization of data, variogram was computed. Suitable model for fitness on experimental variogram was selected based on less Root Mean Square Error (RMSE) value. Then the best method for interpolation was selected, using cross validation and RMSE. Results showed that for all groundwater quality, Ordinary Kriging performed better than IDW method to simulate groundwater quality. Finally, using Ordinary Kriging method, maps of groundwater quality were prepared for studied groundwater quality in Chunnakam aquifer. The result of Ordinary Kriging interpolation showed that higher EC, chloride, sulphate and sodium concentrations are clearly shown to be more common closer to the coast, and decreasing inland due to intrusion of seawater into the Chunnakam aquifer. Also higher $NO_3^- - N$ are observed in intensified agricultural areas of Chunnakam aquifer in Jaffna Peninsula.

Keywords: Chunnakam aquifer, groundwater quality, geostatistics, interpolation, IDW, Ordinary Kriging.

1 Introduction

The usage of groundwater has gradually increased due to the increase of water demand during the growth of population and rapid industrialization. Groundwater can become contaminated from numerous types of human activities such as agricultural, residential, municipal, commercial and industrial usage (Nas, 2009).The Jaffna Peninsula lies in the Northern most part of Sri Lanka. The Jaffna Peninsula has four main types of aquifer

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system such as Chunnakam, Thenmaradchi, Vadamaradchi and Kayts (Punthakey and Nimal, 2006). The water resources of the Valigamam region (Chunnakam aquifer) of the Jaffna Peninsula depend totally on rainfall recharge to the Miocene limestone aquifer (Rajasooriyar *et al*., 2002). Fertilizer and pesticide residues leached from agricultural fields often contribute significantly to groundwater pollution in Jaffna Peninsula. Pollution of groundwater by nitrate as nitrogen has been receiving attention in the Peninsula since early 1980s (Maheswaran and Mahalingam, 1983; Dissanayake and Weerasooriya, 1985; Nagarajah *et al*., 1988; Maheswaran, 2003; Mikunthan and Silva, 2008). Determination of water quality is important because of it helps to protect, restore the quality of the groundwater, and management of groundwater that consistent with the requirements of the clean water.

Geographic Information System (GIS) produced graphical image will provide for user an easier visual inspection of the water quality conditions. Geostatistical method is one of the tools for mapping of groundwater quality. The most common interpolation techniques calculate the estimates for a property at any given location by a weighted average of nearby data. Weighting is assigned either according to deterministic or geostatistical criteria. Among geostatistical methods, kriging based techniques, including simple and Ordinary Kriging (OK), universal kriging and simple cokriging have been often used for spatial analysis (Deutsch, 2002). Among deterministic interpolation methods, Inverse Distance Weighting (IDW) method and its modifications are often applied (Nalder and Wein, 1998). Kriging and IDW are the most commonly used methods in agriculture practices (Franzen and Peck, 1995). Kriging is a method of interpolation, which predicts unknown values from data observed at known locations, and it minimizes the error of predicted values, which are estimated by spatial distribution of the predicted values (Huang *et al*., 2012). Kriging requires the preliminary modeling step of a variance-distance relationship, but IDW does not require such step and is very simple and quick technique for interpolation (Jafar *et al*., 2009).

In recent years, many scientists have evaluated accuracy of different spatial interpolation methods for prediction of soil and water quality parameters. Fahid *et al*. (2011) performed IDW, Kriging, Spline techniques for predicting chloride concentration and groundwater level in Gaza Strip, which showed that Kriging method, provide results that are more accurate. Abdolrahim *et al*. (2011) said that for the estimation of SAR and chloride of groundwater in Iran, the Cokriging method was more accurate than Kriging method. Jafar *et al*. (2009) studied to determine degree of spatial variability of soil chemical properties with Ordinary Kriging and IDW methods, which showed that the Ordinary Kriging performed much better than the IDW in Fars province, Iran. The Ordinary Kriging method was used to produce the spatial patterns of important water quality in Turkey by Nas (2009). The effect of interpolation methods on the accuracy of the GIS mapping was also recognized by Mehrjardi *et al.,* 2008. They compared the efficiency of three interpolation techniques such as IDW, Kriging and Cokriging for predicting some groundwater quality indices in Azarbayjan Province, Iran. The results showed that Cokriging performed better than the other methods. Also Mehrjardi *et al.* (2008) compared above three interpolation techniques for predicting some other groundwater quality characters in Yazd-Ardakan Plain. Results showed that Kriging and Cokriging methods are superior to IDW method.

Therefore, the present study was carried out to select best-suited method to evaluate accuracy of different interpolation methods, Ordinary Kriging and IDW, for prediction of some groundwater quality parameters of Chunnakam aquifer in Jaffna Peninsula.

2 Materials and methods

2.1 Description of the studied area

Valikamam area, which is covered by Chunnakam aquifer, is an intensified agricultural and high population density area in Jaffna Peninsula. The major rainy season in the Peninsula occurs from October to December and the minor rainy season occurs from April and May. The period between South-West monsoon and the North-East monsoon is the dry season and extends from June to September. The major soils are the calcic red-yellow latosols which are shallow, fine textured and well-drained with very rapid infiltration rate (De Alwis and Panabokke, 1972). Agriculture is the main source of livelihood for 65 % of the population and about 34.2 % of the land is cultivated intensively and commercially with high value cash crops (Thadchagini and Thirudchelvam, 2005).

2.2 Selection of wells

Forty four wells were selected for long term water quality monitoring in a systematic manner to represent the entire Chunnakam aquifer. All selected wells are under multiple usages such as domestic wells, wells with domestic and home gardening, public wells for drinking purpose, and farm wells. ***Figure 1*** shows the locations of the wells selected for monitoring in Chunnakam aquifer of the Valikamam area in Jaffna Peninsula.

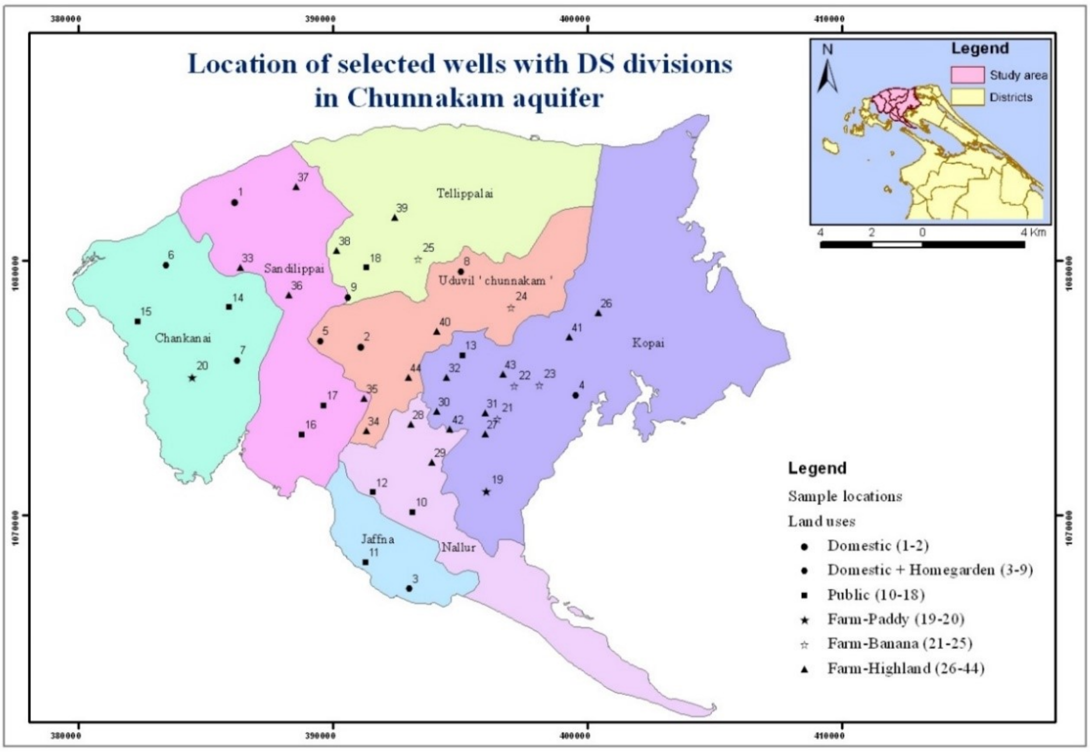

Figure 1: *Location of selected wells with different land usage in Chunnakam aquifer*

2.3 Collection of water sample and analytical techniques

Water samples were collected for chemical analysis five times during mid of January, early part of March, mid of April, mid of July and mid of October 2011 to represent various rainfall regimes within a year. Samples were analyzed for Electrical Conductivity (EC), pH, nitrate as nitrogen, chloride, calcium, carbonate, bicarbonate, sulfate and sodium concentration. Conductivity meter and pH meter were used to measure the pH and EC respectively. Nitrate-N concentration was estimated using colorimetric spectrophotometer. Chloride concentration was measured by silver nitrate titration. Calcium content was determined by EDTA titration using Eriochrome black T as indicator. Carbonate and bicarbonate content were measured by acid-base titration. Sulfate content was estimated by turbidimetric method using turbidity meter. Sodium content was determined by using a flame photometer in Institute of Fundamental Studies (IFS), Hantana, Kandy. The procedures of the analysis were based on Sri Lankan Standard 614 (SLS, 1983).

2.4 Geostatistical approach for spatial prediction of groundwater quality

After normalization of data, for interpolation of groundwater quality, Ordinary Kriging and IDW methods were used. With the use of cross validation, the best method of interpolation was selected. The maps of groundwater quality were prepared based on Ordinary Kriging and IDW interpolation method using ArcGIS 10. Geospatial techniques; Gradient analysis and local indicators of spatial autocorrelations were used to study the groundwater quality and availability assessment for the sustainable management of groundwater in the coastal areas of Jaffna Peninsula (Gunalan *et al.*, 2018).

2.5 Spatial prediction methods

2.5.1 Ordinary Kriging

The presence of a spatial structure where observations close to each other are more alike than those that are far apart (spatial autocorrelation) is a prerequisite to the application of geostatistics (Robinson and Metternicht, 2006). Variogram is used to describe the spatial structure of variable. The variogram of samples, which is also called experimental variogram, measures the average degree of dissimilarity between un-sampled values and a nearby data value and thus can depict autocorrelation at various distances. The value of the experimental variogram for a separation distance of h (referred to as the lag) is half the average squared difference between the value at $z(x_i)$ and the value at $z(x_i+h)$ (Robinson and Metternicht, 2006):

$$\gamma(h) = \frac{1}{2N(h)} \sum_{i=1}^{N(h)} [z(x_i + h) - z(x_i)]^2 \quad \text{.....(1)}$$

where N(h) is the number of data pairs within a given class of distance and direction. If the values at $z(x_i)$ and $z(x_i + h)$ are auto correlated the result of ***Equation (1)*** will be small, relative to an uncorrelated pair of points. From analysis of the experimental variogram, a suitable model (circular, spherical, exponential and guassian) is then fitted, usually by weighted least squares and the parameters (nugget, sill and range) are then used in the Ordinary Kriging procedure.

2.5.2 Inverse Distance Weighting (IDW)

In interpolation with IDW method, a weight is attributed to the point to be measured. The amount of this weight is depended to the distance of the point to another unknown point. These weights are controlled on the bases of power of ten. With increase of power of ten, the effect of the points that are farther diminishes. Lesser power distributes the weights more uniformly between neighbouring points. In IDW, the distance between the points count, so the points of equal distance have equal weights (Burrough and McDonnell, 1998). In this method, the weight factor is calculated with the use of the following formula ***Equation (2)***:

$$\lambda_i = \frac{D_i^{-\alpha}}{\sum_{i=1}^{n} D_i^{-\alpha}} \quad \text{(2)}$$

where λi = the weight of point,
Di = the distance between point i and the unknown point,
a = the power ten of weight

2.6 Comparison between the different methods:

Finally, criterion of Root Mean Square Error (RMSE) is used to evaluate model performances in cross validation mode. The smallest RMSE indicate the most accurate predictions. The RMSE was derived according to ***Equation (3)***.

$$\text{RMSE}=\sqrt{1/N\sum_{i=1}^{N}(Z(x_i)-\ Z^*(x_i)^2} \qquad (3)$$

$Z(x_i)$ is observed value at point xi, $Z^*(x_i)$ is predicted value at point xi, N is number of samples.

3 Result and discussion

A statistical summary of the groundwater quality properties is presented in ***Table 1***. The EC of the water samples is an indicator of their salinity. The values of EC ranged from 556 to 4701µS/cm, with a mean of 1534.31 µS/cm. This behavioural response was used to determine the nature of salinity in studied area. The results revealed that pH ranged from 7.12 to 8.25. All groundwater samples were found to be below the desirable level of Sri Lankan Standard (SLS) for drinking water of pH (7 – 8.5), with a mean of 7.52 and slight alkalinity in nature. The nitrate as N concentration was ranged from 0.28 to 13.86 mg/L. The chloride concentrations of water samples were between 153.86 mg/L to 1145.95 mg/L and mean value is 327.39 mg/L. All values of measured wells were below the permissible level of SLS (1200 mg/L) for drinking. Based on the chloride concentrations all the wells were suited for drinking. The concentration of calcium values of selected wells varied from 58.90 mg/L to 203.06 mg/L and all measured wells were below the permissible level of SLS which is 240 mg/L for drinking water. The concentration (mg/L) of other ions varied as CO_3^{2-} from 11.73 to 61.37; HCO_3^- 158.45 to 545.75; SO_4^{2-} 35.16 to 499.46 and Na^+ 17.71 to 763.20.Results showed that the majority of studied parameters had high skewness, due to insufficient number of samples and unsuitable distribution. However, data were normalized using logarithmic method ***(Table 1).***

Table 1: *Results of statistical analysis on groundwater quality*

Groundwater quality	Minimum	Maximum	Mean	Std. dev.	Kurtosis	Skewness
EC	556	4701	1534.3	1045.7	5.4416	1.8551
EC**	6.32	8.45	7.17	0.53	3.268	0.9435
pH	7.198	8.25	7.52	0.173	8.326	1.5798
pH**	1.973	2.111	2.018	0.022	7.622	1.422
$NO_3^- - N$	0.28	13.86	4.86	3.99	2.507	0.805
$NO_3^- - N$**	-1.273	2.629	1.139	1.0782	2.4067	-0.586
Cl^-	153.86	1145.9	327.39	246.21	6.8104	2.2182
Cl^-**	5.036	7.044	5.62	0.527	4.181	1.4076
Ca^{2+}	58.9	203.06	95.46	35.72	5.33	1.6891
Ca^{2+}**	4.07	5.31	4.504	0.317	3.478	0.9954
CO_3^{2-}	11.73	61.366	28.18	9.5122	4.656	0.7781
CO_3^{2-}**	2.4623	4.1169	3.28	0.3461	2.867	-0.31126
HCO_3^-	158.45	545.75	258.68	90.97	4.555	1.4609
HCO_3^-**	5.065	6.3022	5.5055	0.3073	3.028	0.8655
SO_4^{2-}	35.161	499.46	150.99	106.71	4.506	1.445
SO_4^{2-}**	3.55	6.2135	4.8068	0.6477	2.4464	0.2337
Na^+	17.708	763.2	149.32	171.96	6.8354	2.1071
Na^+**	2.874	6.6375	4.544	0.9234	2.6243	0.5531

Except pH and EC (µS/cm), the others parameters are expressed in mg/L.
**Using logarithm to normalize data

After data normalizing, experimental variogram was computed. In this study, the variogram models (circular, spherical, exponential and guassian) were tested for each parameter in groundwater quality. Prediction performances were assessed by cross validation, which examines the accuracy of the generated surfaces. The best model for fitting on experimental variogram was selected based on less RMSE value (***Table 2***).Circular and exponential model are selected for EC and Ca^{2+} respectively. Spherical model is selected for $NO_3^- - N$, SO_4^{2-} and Na^+. Guassian model is selected for pH, Cl^-, CO_3^{2-} and HCO_3^- in geostatistic analysis.

Table 2: *Selection of the most suitable model for evaluation on experimental variogram according to RMSE*

Groundwater quality	Models			
	Circular	**Spherical**	**Exponential**	**Guassian**
EC	**782.6545**	783.0075	792.9405	784.9210
pH	0.1760	0.1763	0.1803	**0.1751**
$NO_3^- - N$	3.4702	**3.4488**	3.4672	3.4812
Cl^-	157.5719	157.5118	160.1213	**156.7297**
Ca^{2+}	31.6084	31.6577	**31.5644**	31.7122
CO_3^{2-}	8.5052	8.5046	8.5031	**8.5000**
HCO_3^-	73.6457	73.6817	74.2180	**73.5617**
SO_4^{2-}	88.2393	**88.2262**	90.2392	88.6274
Na^+	120.6330	**120.5626**	121.2188	122.2146

Table 3: *Best fitted variogram models of groundwater quality and their parameters*

Groundwater quality	Model	Nugget (C_o)	Sill (C_o+C)	Range effect (km)	(C_o/C_o+C) %
EC	Circular	0.01871	0.22860	5.21	8
pH	Guassian	0.00025	0.00033	4.58	76
$NO_3^- - N$	Spherical	0.25110	0.51510	2.26	49
Cl^-	Guassian	0.04838	0.21480	6.76	23
Ca^{2+}	Exponential	0.00000	0.09550	4.72	0
CO_3^{2-}	Guassian	0.00560	0.11110	5.06	5
HCO_3^-	Guassian	0.02031	0.07660	6.81	27
SO_4^{2-}	Spherical	0.02254	0.41070	5.25	5
Na^+	Spherical	0.10320	0.78260	7.24	13

Also, ***Table 3*** illustrates parameters of groundwater quality variograms. The ratio of nugget variance to sill expressed in percentages can be regarded as a criterion for classifying the spatial dependence of groundwater quality parameters. If this ratio is less than 25%, then the variable has strong spatial dependence; if the ratio is between 25 and 75%, the variable has moderate spatial dependence; and greater than 75%, the variables shows only weak spatial dependence (Jiachun *et al.*, 2007).Some parameters of groundwater quality such as EC, Cl^-, Ca^{2+}, CO_3^{2-}, SO_4^{2-} and Na^+ have strong spatial dependence due to the effect of natural factors including sea water intrusion and water-soil/rock interaction. NO_3^- as N and HCO_3^- in groundwater quality have moderate spatial dependence, indicating an involvement of human factors and pH has weak spatial dependence. Also effective range of most parameters is close together with the range of 2.26 km to 7.24 km.

Table 4: *Selecting the best power according to RMSE in IDW method*

Groundwater quality	Power		
	1	2	3
EC	933.2	853.0	**783.9**
pH	0.180	0.179	**0.178**
$NO_3^- - N$	3.560	**3.505**	3.553
Cl^-	207.3	183.1	**162.2**
Ca^{2+}	33.110	31.898	**31.569**
CO_3^{2-}	**8.97**	9.21	9.64
HCO_3^-	80.40	76.70	**74.10**
SO_4^{2-}	101.80	96.80	**92.55**
Na^+	147.40	133.18	**119.96**

IDW predictions were performed varying the number of power (from 1-3). The results, in terms of RMSE, obtained from the cross validation procedures are presented in ***Table 4***. The RMSE are generally lower for IDW with power of 3 in comparison to that of other powers for most of the groundwater quality parameters such as EC, pH, Cl^-, Ca^{2+}, HCO_3^-, SO_4^{2-} and Na^+. Power 1 and 2 are fitted for CO_3^{2-} and $NO_3^- - N$ respectively.

RMSE, for determination of the most suitable method, among Ordinary Kriging and IDW, was used. Based on ***Table 5***, Ordinary Kriging method increased prediction accuracy and had less RMSE for all studied parameters in Chunnakam aquifer. Results showed that Ordinary Kriging method is best GIS interpolation method for predicting all studied parameters of Chunnakam aquifer in Jaffna Peninsula. Karami *et al.*, 2018 showed that assessment of groundwater resources through the ordinary kriging is an appropriate method for estimating the values and producing reliable data and increasing the accuracy of assessment. Also study showed that geostatistics is a powerful tool to determine subsurface heterogeneity for hydrogeological applications in a wide range of complex geological environment by applying geostatistics tool to a real aquifer.

Table 5: *Selecting the best interpolation method according to RMSE*

Groundwater quality	IDW	Ordinary Kriging
EC	783.9	**782.7**
pH	0.178	**0.175**
$NO_3^- - N$	3.505	**3.449**
Cl^-	162.2	**156.7**
Ca^{2+}	31.569	**31.564**
CO_3^{2-}	8.97	**8.50**
HCO_3^-	74.10	**73.56**
SO_4^{2-}	92.55	**88.23**
Na^+	119.96	**119.56**

Finally, maps of groundwater quality were prepared using Ordinary Kriging which was the best method for interpolation in Chunnakam aquifer. ***Figure 2- 10*** shows the spatial distribution of studied groundwater quality parameters using Ordinary Kriging and IDW in Chunnakam aquifer. Based on these figures, Ordinary Kriging avoids the “bulls eye” effect. The spatial distribution of EC in Chunnakam aquifer is shown in ***Figure 2.*** Higher EC was clearly shown to be more common closer to the coast, and decreasing inland of Chunnakam aquifer in Jaffna Peninsula. The above results are in agreement with the spatial distribution of chloride, sodium and sulphate (***Figure 3, 4*** and ***5*** respectively). The concentration of Na^+, Cl^- and SO_4^{2-} in seawater is much greater than in continental water. This distribution pattern can be ascribed to the intrusion of seawater into the aquifer system which increases the concentrations of these ions and hence values of the dissolved solids. The trend of EC generally reflects the chloride concentration available in groundwater and enriched by the discharge ions of sodium, calcium and magnesium (Jothivenkatachalam *et al.*, 2011).

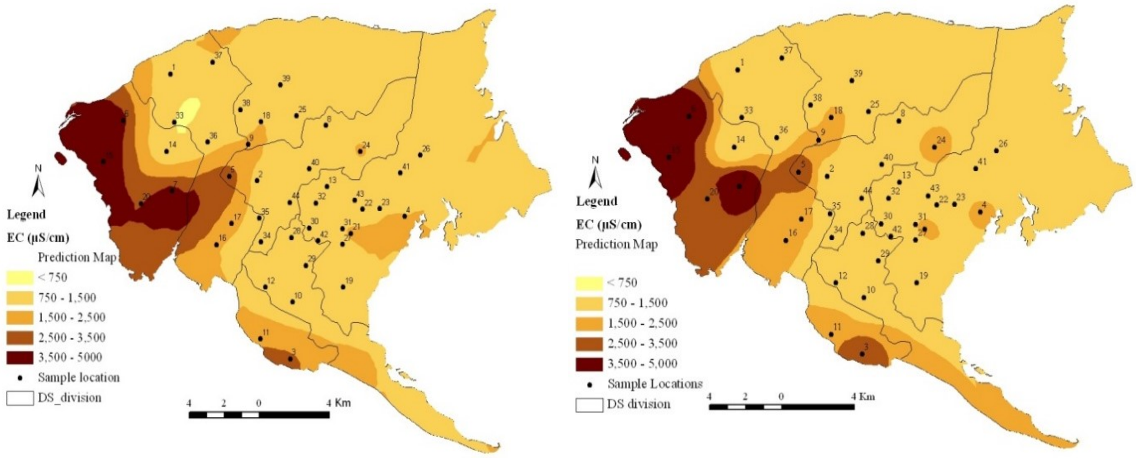

Figure 2 : *Spatial distribution of EC based on (a) Ordinary Kriging and (b) IDW methods*

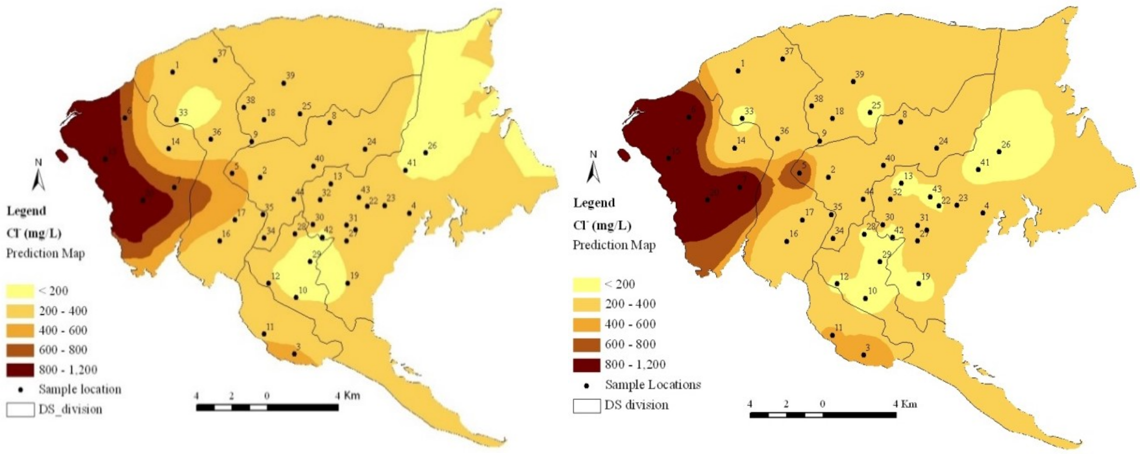

Figure 3: *Spatial distribution of Cl^{-} based on (a) Ordinary Kriging and (b) IDW methods*

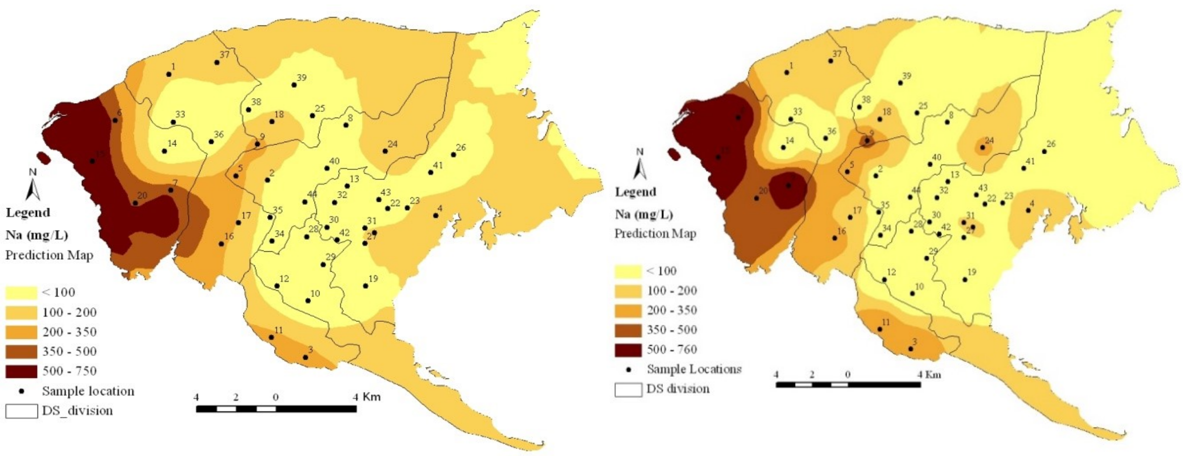

Figure 4: *Spatial distribution of Na based on (a) Ordinary Kriging and (b) IDW methods*

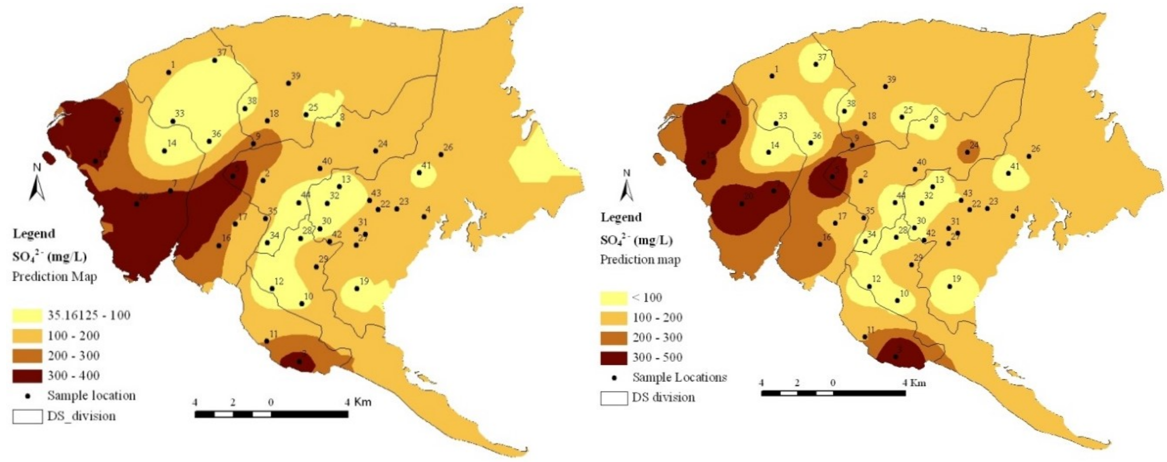

Figure 5: *Spatial distribution of* $\mathbf{SO_4^{2-}}$ *based on (a) Ordinary Kriging and (b) IDW methods*

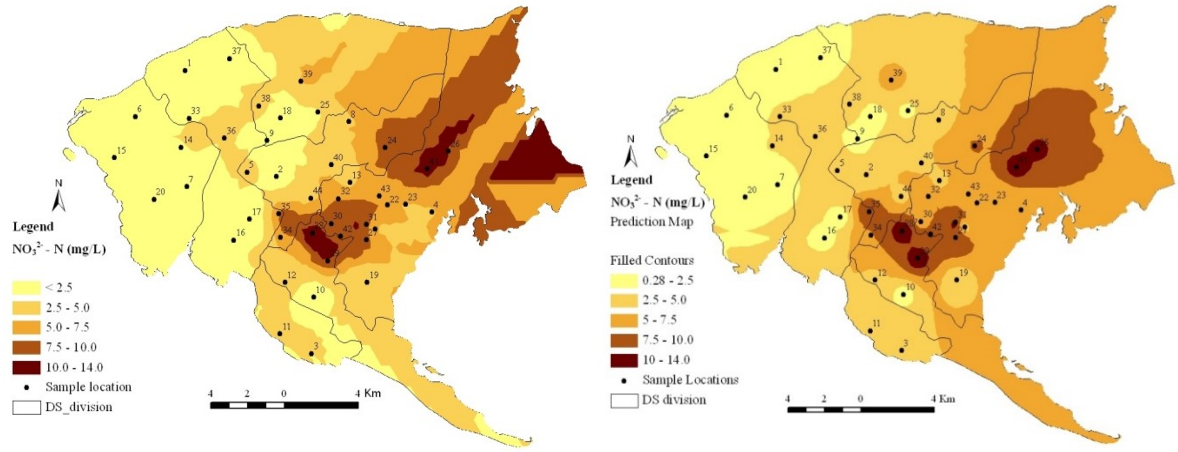

Figure 6: *Spatial distribution of* $\boldsymbol{NO_3^-} - \boldsymbol{N}$ *based on (a) Ordinary Kriging and (b) IDW methods*

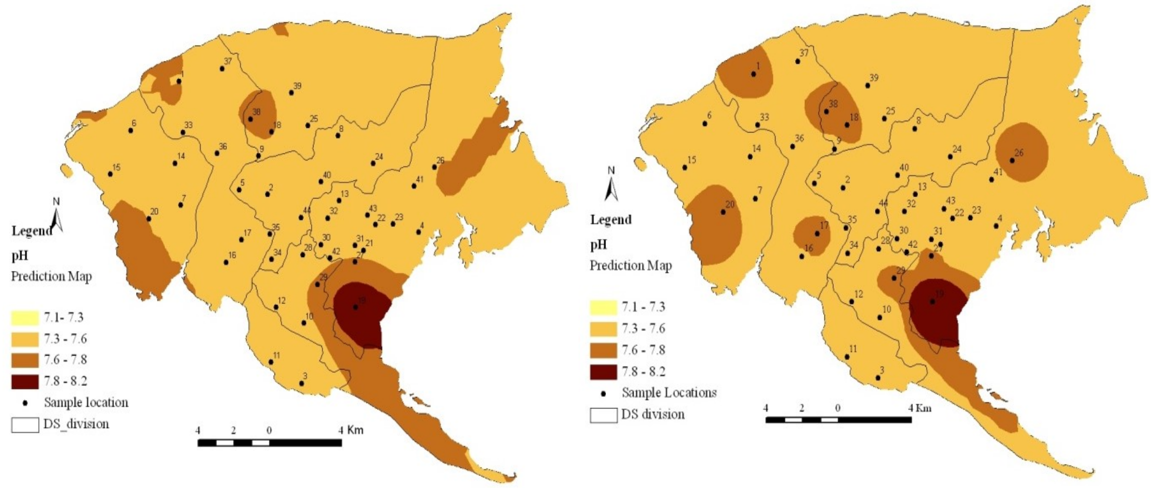

Figure 7: *Spatial distribution of pH based on (a) Ordinary Kriging and (b) IDW methods*

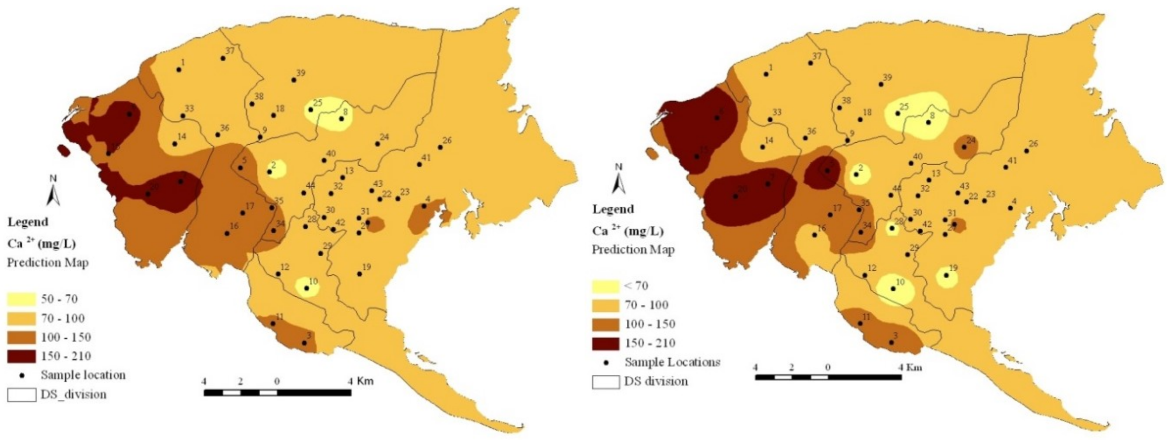

Figure 8: *Spatial distribution of Ca^{2+} based on (a) Ordinary Kriging and (b) IDW methods*

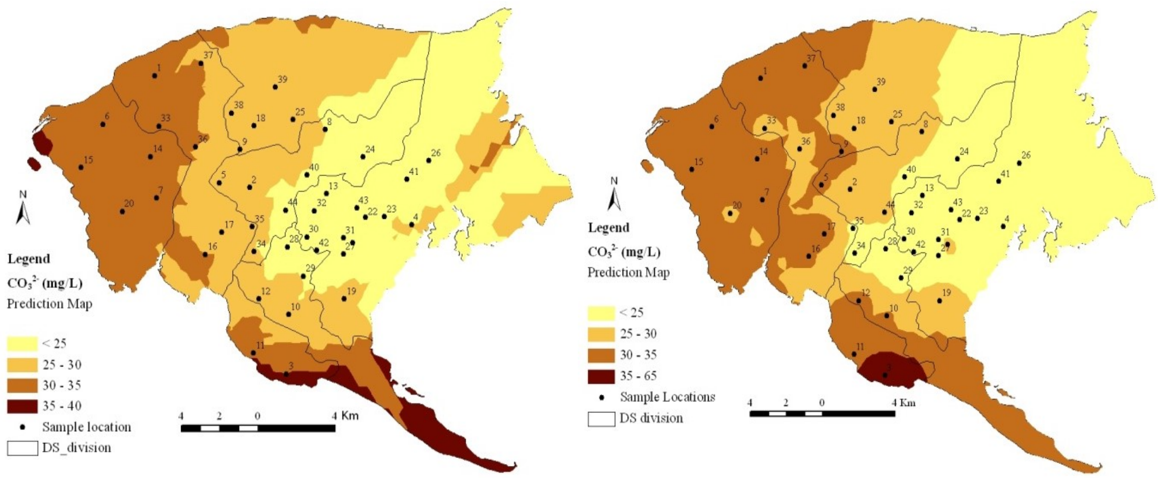

***Figure 9**:* *Spatial distribution of $\boldsymbol{CO_3^{2-}}$ based on (a) Ordinary Kriging and (b) IDW methods*

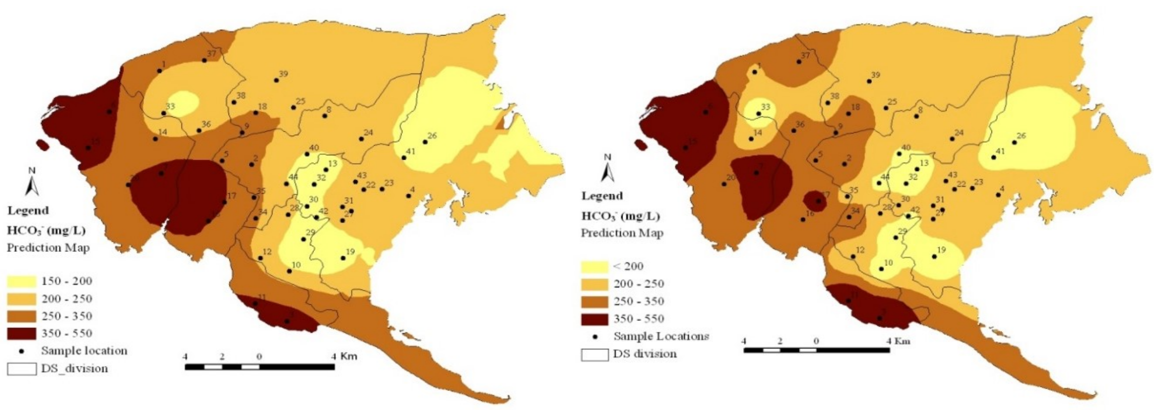

Figure 10: *Spatial distribution of $\boldsymbol{HCO_3^-}$ based on (a) Ordinary Kriging and (b) IDW methods*

Based on NO_3^- as N, intensified agricultural areas of Chunnakam aquifer have above permissible level of SLS for drinking which is showed in *Figure 6*. Gunasegaram (1983) studied extensively groundwater contamination in the Jaffna Peninsula and found that the nitrate levels exceeded standard limits, which is due to the mixing up of abundant nitrogenous waste matter and synthetic and animal fertilizers reaching the shallow groundwater table. Dissanayake and Weerasooriya (1985) pointed out in hydro geochemical atlas of Sri Lanka that Jaffna Peninsula has the highest nitrate content among the groundwater of Sri Lanka.

Salinity development and high concentrations of nitrate–N were the identified problems in Chunnakam aquifer of Jaffna Peninsula.

4 Conclusion

This study has attempted to predict the spatial distribution and uncertainty of some groundwater quality in Chunnakam aquifer, Jaffna Peninsula, using two interpolation techniques (Ordinary Kriging and IDW).The majority of studied parameters had high skewness. The analysis showed that for all groundwater quality Ordinary Kriging performed better than IDW techniques in characterizing the spatial variability.The result of Ordinary Kriging interpolation showed that development of salinity is clearly shown to be more common closer to the coast, and decreasing inland and higher $NO_3^- - N$ also is observed in intensified agricultural areas of Chunnakam aquifer in Jaffna Peninsula. It is suggested that in the future studies, other methods especially indicator and disjunctive kriging is used in order to prepare risk maps of Chunnakam aquifer.

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Research Article

Frame Multipliers for Discrete Frames on Quaternionic Hilbert Spaces

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Abstract:

In this note, following the complex theory, we examine discrete controlled frames, discrete weighted frames and frame multipliers in a non-commutative setting, namely in a left quaternionic Hilbert space. In particular, we show that the controlled frames are equivalent to usual frames under certain conditions. We also study connection between frame multipliers and weighted frames in the same Hilbert space.

Keywords: Quaternions, Quaternion Hilbert spaces, Frames, Frame Multipliers

1 Introduction

The notion of frames in Hilbert spaces was introduced by Duffin and Schaeffer in 1952 to address some very deep problems in non-harmonic Fourier series (Duffin & Schaeffer, 1952). However the fundamental concept of frames was revived in 1980s by Daubechies, Grossmann and Meyer (Daubechies, 1992) (Daubechies, Grossmann, & Meyer, 1986), who showed its significance in signal processing. Frame is a spanning set of vectors which are generally overcomplete (redundant) in a Hilbert space. Therefore a typical frame contains more frame vectors than the dimension of the space and each vector in the space will have infinitely many representations with respect to the frame. This redundancy of frames is the key to their success in applications.

Nowadays frames have broad applications in Mathematics and Engineering in several areas including sampling theory (Aldroubi & Grochenig, 2001), operator theory (Han & Larson, 2000), harmonic analysis (Lacey & Thiele, 1997), wavelet theory (Daubechies, 1990), wireless communication (Shrohmer & Beaver, 2003) (Shrohmer & Heath, 2003), data transmission with erasures (Bodmann & Paulsen, 2005) (Goyal, Kovacevic, & Kelner, 2000), filter banks (Bolckei, Hlawatsch, & Feichtinger, 1998), signal processing (Benedetto, Powell, & Yilmaz, 2006), (Goyal, Vetterli, & Nguyen, Quantized overcomplete expansions in Rn : analysis, synthesis, and algorithms, 1998) image processing (Candes & Donoho, 2005)], geophysics (Margrave, et al., 2005) and quantum computing (Eldar & Forney, 2002). Hilbert spaces can be defined over the fields $\mathbb{R}$, the set of all real numbers, $\mathbb{C}$, the set of all complex numbers, and $\mathbb{H}$, the set of all quaternions only (Adler, 1995). The fields $\mathbb{R}$ and $\mathbb{C}$ are associative and commutative but quaternions form a non-commutative associative algebra and this feature highly restricted mathematicians to work out a well-formed theory of functional analysis and

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harmonic analysis on quaternionic Hilbert spaces. The quaternionic frames have been developed in the mathematical point of view very recently (Khokulan, Thirulogasanthar, & Srisatkunarajah, 2017) (Khokulan, Thirulogasanthar, & Muraleetharan, 2015). The applications are yet to be identified and analyzed in terms of these frames.

Frame multiplier is an operator which was introduced by P.Balazs for frames in Hilbert spaces (Balazs, 2007). In these multipliers the analysis coefficients are multiplied by a fixed sequence (called the symbol) before re-synthesis. Fundamental properties of this multiplier were investigated in (Balazs, 2007). Frame multipliers are interesting not only from a theoretical point of view but also for applications. For example, it is useful in psycho-acoustical modeling (Balazs, Laback, Eckel, & Deutsch, 2010), denoising (Majdak, Balazs, Kreuzer, & Dorfler, 2011), computational auditory scene analysis (Wang & Brown, 2006), virtual acoustics (Majdak, Balazs, & Laback, 2007) and seismic data analysis (Margrave, et al., 2005).

The extensions of frames in Hilbert spaces include weighted and controlled frames and these extensions were introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator (Balazs, Antoine, & Grybos, 2010). In this paper we investigate the connection between the weighted frames and frame multipliers for discrete frames in left quaternionic Hilbert spaces along the lines of the argument of P. Balazs *et al.* (Balazs, Antoine, & Grybos, 2010).

This article is organized as follows: In section 2, we collect basic notations and some preliminary results about quaternions and frames as needed for the development of the results obtained in this article. In section 3, we present the concept of controlled frames in quaternionic Hilbert spaces and we will show that controlled frames are equivalent to usual frames under certain conditions. In section 4, we investigate the weighted frames and frame multipliers. We also investigate how the frame multipliers are connected with weighted frames in quaternionic Hilbert spaces. Section 5 ends the article with a conclusion.

2 Mathematical Preliminaries

We recall few facts about quaternions, quaternionic Hilbert spaces and quaternionic functional analysis, which may not be very familiar to the reader.

2.1 Quaternions

Let $\mathbb{H}$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_1, x_1, x_2 and x_3 are real numbers, and *i, j, k* are imaginary units such that $i^2 = j^2 = k^2 = -1, ij = -ji = k,\ jk = -kj = i$ and $ki = -ik = j$.

The quaternionic conjugate of q is defined to be $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$. Quaternions do not commute in general. However, q and $\bar{q}$ commute, and quaternions commute with real numbers. $|q|^2 = q\bar{q} = \bar{q}q \text{ and } \overline{qp} = \bar{p}\,\bar{q}$

2.2 Left Quaternionic Hilbert Space

Let $V_{\mathbb{H}}^{L}$ be a vector space under left multiplication by quaternionic scalars, where $\mathbb{H}$ stands for the quaternion algebra.

For $f, g, h \in V_{\mathbb{H}}^{L}$ and $q \in \mathbb{H}$, the inner product

$$\langle . | . \rangle : V_{\mathbb{H}}^{L} \times V_{\mathbb{H}}^{L} \to \mathbb{H}$$

satisfies the following properties:

(a) $\overline{\langle f|g\rangle} = \langle g|f\rangle >$

(b) $\|f\|^2 = \langle f|f\rangle > 0$ unless $f = 0$, a real norm

(c) $\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle$

(d) $\langle qf|g\rangle = q\langle f|g\rangle$
(e) $\langle f|qg\rangle = \langle f|g\rangle\bar{q}$

where $\bar{q}$ stands for the quaternionic conjugate of q. We assume that the space $V_{\mathbb{H}}^{L}$ is complete under the norm given above. Then, together with $\langle .|.\rangle$ this defines a left quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwartz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals.

Dirac bra-ket notation can be adapted as

$$|q\emptyset\rangle = |\emptyset\rangle\bar{q}, \langle\emptyset| = |\emptyset\rangle^{\dagger}, \langle q\emptyset| = q\langle\emptyset| \qquad \mathbf{(2.1)}$$

for a left quaternionic Hilbert space, with $|\emptyset\rangle$ denoting the vector $\emptyset$ and $\langle\emptyset|$ is its dual vector.

Let T be an operator on a left quaternionic Hilbert space $V_{\mathbb{H}}^{L}$ with domain $D(T)$. The adjoint $T^{\dagger}$ of T is defined as

$$\langle\psi|T\emptyset\rangle = \langle T^{\dagger}\psi|\emptyset\rangle \text{ for all } \psi,\emptyset \in V_{\mathbb{H}}^{L}. \qquad \mathbf{(2.2)}$$

An operator T is said to be self-adjoint if $T = T^{\dagger}$.

Let $D(T)$ denotes the domain of T. The operator T is said to be left linear if

$$T(\emptyset + \psi) = T\emptyset + T\psi,$$
$$T(q\emptyset) = q(T\emptyset)$$

for all $\emptyset,\psi \in D(T)$ and $q \in \mathbb{H}$. The set of all left linear operators in $V_{\mathbb{H}}^{L}$ will be denoted by $L(V_{\mathbb{H}}^{L})$. For a given $T \in L(V_{\mathbb{H}}^{L})$, the range and the kernel will be

$$ran(T) = \{\psi \in V_{\mathbb{H}}^{L} | T\emptyset = \psi \text{ for } \emptyset \in D(T)\}$$
$$ker(T) = \{\emptyset \in D(T) \mid T\emptyset = 0\}.$$

We call an operator $T \in L(V_{\mathbb{H}}^{L})$ is bounded if

$$\|T\| = \sup_{\|\emptyset\|=1} \|T\emptyset\| < \infty$$

or equivalently, there exists $M \geq 0$ such that $\|T\emptyset\| \leq M\|\emptyset\|$ for $\emptyset \in D(T)$. The set of all bounded left linear operators in $V_{\mathbb{H}}^{L}$ will be denoted by $B(V_{\mathbb{H}}^{L})$.

Proposition 2.1. (Fashandi, 2014) Let $T \in B(V_{\mathbb{H}}^{L})$. Then T is self-adjoint if and only if for each

$$\emptyset \in V_{\mathbb{H}}^{L}, \langle T\emptyset|\emptyset\rangle \in \mathbb{R}.$$

Definition 2.2. Let T_1 and T_2 be self-adjoint operators on $V_{\mathbb{H}}^{L}$. Then $T_1 \leq T_2$ (T_1 less or equal to T_2) or equivalently $T_2 \geq T_1$ if $\langle T_1\emptyset|\emptyset\rangle \leq \langle T_2\emptyset|\emptyset\rangle$ for all $\emptyset \in V_{\mathbb{H}}^{L}$. In particular T_1 is called positive if $T_1 \geq 0$ or $\langle T_1\emptyset|\emptyset\rangle \geq 0$, for all $\emptyset \in V_{\mathbb{H}}^{L}$.

Theorem 2.3. (Ghiloni, Moretti, & Perotti, 2013) *Let $A \in B(V_{\mathbb{H}}^{L})$. If $A \geq 0$ then there exists a unique operator in $B(V_{\mathbb{H}}^{L})$, indicated by $\sqrt{A} = A^{1/2}$ such that $\sqrt{A} \geq 0$ and $\sqrt{A}\sqrt{A} = A$.*

Proposition 2.4. (Ghiloni, Moretti, & Perotti, 2013)Let $A \in B(V_{\mathbb{H}}^{L})$. If $A \geq 0$, then A is self-adjoint.

Lemma 2.5. *Let $U_{\mathbb{H}}^{L}$ and $V_{\mathbb{H}}^{L}$ be left quaternion Hilbert spaces. Let $T : D(T) \to V_{\mathbb{H}}^{L}$ be a linear operator with domain $D(T) \subseteq U_{\mathbb{H}}^{L}$ and $ran(T) \subseteq V_{\mathbb{H}}^{L}$, then the inverse $T^{-1} : ran(T) \to D(T)$ exists if and only if $T\emptyset = 0 \Rightarrow \emptyset = 0$.*

Proof. Suppose that $T\emptyset = 0$ implies $\emptyset = 0$. Let $\emptyset_1, \emptyset_2 \in D(T)$ with $T\varphi_1 = T\varphi_2$ Since T is linear,

$$T(\emptyset_1 - \emptyset_2) = T\emptyset_1 - T\emptyset_2 = 0,$$

so that $\emptyset_1 - \emptyset_2 = 0$, by the supposition. Hence $T\emptyset_1 = T\emptyset_2$ implies $\emptyset_1 = \emptyset_2$.

Thereby T is one to one and it follows that T^{-1} exists.

Conversely suppose that T^{-1} exists, then for any $\emptyset_1, \emptyset_2 \in D(T)$,

$$T\emptyset_1 = T\emptyset_2 \Rightarrow \emptyset_1 = \emptyset_2.$$

If we take $\emptyset_2 = 0$ then

$$T\emptyset_1 = T0 = 0 \Rightarrow \emptyset_1 = 0.$$

Hence $T^{-1} : ran(T) \to D(T)$ exists if and only if $T\emptyset = 0$ implies $\emptyset = 0$.

Lemma 2.6. (Fashandi, 2014)*Let* $T \in B(V_{\mathbb{H}}^L)$ *be a self-adjoint operator, then*

$$\|T\| = \sup_{\|\emptyset\|=1} |\langle \emptyset | T\emptyset \rangle| \qquad \textbf{(2.3)}$$

We define $\mathcal{GL}(V_{\mathbb{H}}^L)$, the set of all bounded linear operators in $V_{\mathbb{H}}^L$ with bounded inverse.

$$\mathcal{GL}(V_H^L) = \{T: V_{\mathbb{H}}^L \to V_{\mathbb{H}}^L: T \text{ bounded and } T^{-1} \text{ bounded})\}$$

Also $\mathcal{GL}^+(V_{\mathbb{H}}^L)$ is the set of positive operators in $\mathcal{GL}(V_{\mathbb{H}}^L)$.

Proposition 2.7. Let $T: V_{\mathbb{H}}^L \to V_{\mathbb{H}}^L$ be a bounded left linear operator. Then the following are equivalent statements:

- i There exist $m > 0$ and $M < \infty$ such that $mI_{V_{\mathbb{H}}^L} \le T \le MI_{V_{\mathbb{H}}^L}$;
- ii T is positive and there exist $m > 0$ and $M < \infty$ such that $m\|f\|^2 \le \left\|T^{\frac{1}{2}}f\right\|^2 \le M\|f\|^2$;
- iii T is positive and $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^L)$;
- iv There exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^L)$ such that $A^2 = T$;
- iv. $T \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$

Proof. For i.⇒ii., suppose that $m > 0$ and $M < \infty$ such that $mI_{V_{\mathbb{H}}^L} \le T \le MV_{\mathbb{H}}^L$. Let $f \in V_{\mathbb{H}}^L$ then $\left\langle mI_{V_{\mathbb{H}}^L} f \middle| f \right\rangle \le \langle Tf|f \rangle \le \left\langle MI_{V_{\mathbb{H}}^L} f \middle| f \right\rangle$. It follows that

$$m\|f\|^2 \le \langle Tf|f \rangle \le M\|f\|^2 \qquad \textbf{(2.4)}$$

Hence $\langle Tf|f \rangle \ge m\|f\|^2 \ge 0$, as $m > 0$. Thereby $\langle Tf|f \rangle \ge 0$ and T is positive. Since T is positive, from Theorem 2.3, there exists a unique operator in $B(V_{\mathbb{H}}^L)$, indicated by $\sqrt{T} = T^{\frac{1}{2}}$ such that $\sqrt{T} \ge 0$ and $\sqrt{T}\sqrt{T} = T$.

Now equation (2.4) becomes

$$m\|f\|^2 \le \left\langle T^{\frac{1}{2}}T^{\frac{1}{2}}f \middle| f \right\rangle \le M\|f\|^2$$

It follows that

$$m\|f\|^2 \le \left\langle T^{\frac{1}{2}}f \middle| \left(T^{\frac{1}{2}}\right)^\dagger f \right\rangle \le M\|f\|^2$$

and

$$m\|f\|^2 \le \left\langle T^{\frac{1}{2}}f \middle| T^{\frac{1}{2}}f \right\rangle \le M\|f\|^2, \text{ as } T^{\frac{1}{2}} \text{ is positive.}$$

Thereby $m\|f\|^2 \le \left\|T^{\frac{1}{2}}f\right\|^2 \le M\|f\|^2$.

For $ii. \Rightarrow i.$, suppose that T is positive and there exists $m > 0$ and $M < \infty$ such that $m\|f\|^2 \le \left\|T^{\frac{1}{2}}f\right\|^2 \le M\|f\|^2$. Then

$$m\|f\|^2 \le \langle T^{\frac{1}{2}}f | T^{\frac{1}{2}}f \rangle \le M\|f\|^2 .$$

It follows that

$$m\|f\|^2 \le \langle \left(T^{\frac{1}{2}}\right)^\dagger T^{\frac{1}{2}}f | f \rangle \le M\|f\|^2$$

and

$m\|f\|^2 \le \langle T^{\frac{1}{2}}T^{\frac{1}{2}}f|f \rangle \le M\|f\|^2$ as $T^{\frac{1}{2}}$ is positive.

Hence

$$m\|f\|^2 \le \langle Tf|f \rangle \le M\|f\|^2 .$$

Therefore $I_{V_{\mathbb{H}}^L} \le T \le MI_{V_{\mathbb{H}}^L}$.

For $ii. \Rightarrow iii.$, suppose that T is positive and there exists $m > 0$ and $M < \infty$ such that $m\|f\|^2 \le \left\|T^{\frac{1}{2}}f\right\|^2 \le M\|f\|^2$. Since T is positive, from Theorem 2.3, $T^{\frac{1}{2}}$is bounded. Let $f \in V_{\mathbb{H}}^L$, assume that $T^{\frac{1}{2}}f = 0$ then $m\|f\|^2 \le \|0\|^2 \le M\|f\|^2$. It follows that $f = 0$, from Lemma 2.5,

$\left(T^{\frac{1}{2}}\right)^{-1}: V_{\mathbb{H}}^{L} \to V_{\mathbb{H}}^{L}$ exists. For $f \in V_{\mathbb{H}}^{L}$, there exists $g \in V_{\mathbb{H}}^{L}$ such that $T^{\frac{1}{2}}f = g$. That is $f = \left(T^{\frac{1}{2}}\right)^{-1} g$. Now $m\|f\|^2 \leq \left\|T^{\frac{1}{2}}f\right\|^2$ implies

$$m\left\|\left(T^{\frac{1}{2}}\right)^{-1} g\right\|^2 \leq \|g\|^2$$

and

$$\left\|\left(T^{\frac{1}{2}}\right)^{-1} g\right\| \leq \frac{1}{\sqrt{m}}\|g\|$$

It follows that $\left(T^{\frac{1}{2}}\right)^{-1}$ is bounded and hence $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$.

For $iii. \Rightarrow\ ii.$, suppose that T is positive and $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$. Then $T^{\frac{1}{2}}$ is bonded and $\left(T^{\frac{1}{2}}\right)^{-1}$ is also bounded. Therefore, one may conclude that there exists $m > 0$ and $M < \infty$ such that $m\|f\|^2 \leq \left\|T^{\frac{1}{2}}f\right\|^2 \leq M\|f\|^2$.

For $iii. \Rightarrow\ iv.$, assume that T is positive and $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$. Since T is positive, from Theorem 2.3, there exists a unique operator in $\mathrm{B}(V_{\mathbb{H}}^{L})$, indicated by $\sqrt{T} = T^{\frac{1}{2}}$ such that $\sqrt{T} \geq 0$ and $(\sqrt{T})^2 = T$. If we take $A = T^{\frac{1}{2}}$ then A is self adjoint as $\sqrt{T} \geq 0$ and $A \in \mathcal{GL}(V_{\mathbb{H}}^{L})$. Hence there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ such that $A^2 = T$.

For iv.⇒ iii., suppose that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ such that $A^2 = T$. If we take $A = T^{1/2}$ then $T^{\frac{1}{2}} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$n) and T is positive.

For $iv. \Rightarrow\ v.$, assume that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ such that $A^2 = T$. Then clearly $T \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L})$.

For $v. \Rightarrow\ iv.$, assume that $T \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L})$. Then T is positive and bounded. From Theorem 2.3, there exist a unique operator $A\left(= T^{\frac{1}{2}}\right)$ in $\mathcal{GL}(V_{\mathbb{H}}^{L})$ such that $A \geq 0$ and $A.A = T$. It follows that there exists a self-adjoint operator $A \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ such that $A^2 = T$.

2.3 Frames and Frame operators

Let $V_{\mathbb{H}}^{L}$ be a finite dimensional left quaternion Hilbert space. A countable family of elements $\{\emptyset_k\}_{k\in I^I}$ in $V_{\mathbb{H}}^{L}$ is a frame for $V_{\mathbb{H}}^{L}$ if there exist constants $A,B > 0$ such that

$$A\|\psi\|^2 \leq \sum_{k\in I}|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2, \text{ for all } \psi \in V_{\mathbb{H}}^{L} \quad \textbf{(2.5)}$$

The numbers A and B are called frame bounds. They are not unique. The *optimal lower frame bound* is the supremum over all lower frame bounds, and the *optimal upper frame bound* is the infimum over all upper frame bounds. The frame $\{\emptyset_k\}_{k\in I}$ is said to be normalized if $\|\emptyset_k\| = 1$, for all $k \in I$.

Let $\{\emptyset_k\}_{k\in I}$ be a frame on a left quaternionic Hilbert space $V_{\mathbb{H}}^{L}$ and define a linear mapping

$$T: \mathbb{H}^{|I|} \to V_{\mathbb{H}}^{L}, T\{q_k\}_{k\in I} = \sum_{k\in I} q_k\emptyset_k\ ,\ \ q_k \in \mathbb{H} \quad \textbf{(2.6)}$$

where $|I|$ is the cardinality of I. T is usually called the *pre-frame operator*, or the *synthesis operator.*

The adjoint operator

$$T^{\dagger}: V_{\mathbb{H}}^{L} \to \mathbb{H}^{|I|}, \text{given by } T^{\dagger}\psi = \{\langle\psi|\emptyset_k\rangle\}_{k\in I} \quad \textbf{(2.7)}$$

is called the *analysis operator.*

By composing T with its adjoint we obtain the *frame operator*

$$S: V_{\mathbb{H}}^{L} \to V_{\mathbb{H}}^{L}, S\psi = TT^{\dagger}\psi = \sum_{k\in I}\langle\psi|\emptyset_k\rangle\emptyset_k \quad \textbf{(2.8)}$$

Note that in terms of the frame operator, for $\psi \in V_{\mathbb{H}}^{L}$

$$\langle S\psi|\psi\rangle = \langle \sum_{k\in I}\langle\psi|\emptyset_k\rangle\, \emptyset_k|\psi\rangle$$
$$= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\langle\emptyset_k|\psi\rangle$$
$$= \sum_{k\in I}|\langle\psi|\emptyset_k\rangle|^2$$

Proposition 2.8. (Khokulan, Thirulogasanthar, & Srisatkunarajah, Discrete frames on finite dimensional quaternion Hilbert spaces, 2017) Let $\{\emptyset_k\}_{k\in I}$ be a frame for $V_{\mathbb{H}}^L$ with frame operator *S.* Then

i S is invertible and self-adjoint.

ii Every $\psi \in V_{\mathbb{H}}^L$, can be represented as

$$\psi = \sum_{k\in I}\langle\psi|S^{-1}\emptyset_k\rangle\emptyset_k = \sum_{k\in I}\langle\psi|\emptyset_k\rangle S^{-1}\emptyset_k$$

3 Controlled Frames in Left Quaternion Hilbert Spaces

Controlled frames were first introduced for spherical wavelets in (Bogdanova, Vandergheynst, Antoine, Jacques, & Morvidone, 2005) to get a numerically more efficient approximation algorithm. For general frames, it was developed in (Balazs, Antoine, & Grybos, Weighted and controlled frames: mutual relationship and first numerical properties, 2010). In this section the concept of controlled frames in quaternionic Hilbert spaces, along the lines of the arguments of (Balazs, Antoine, & Grybos, Weighted and controlled frames: mutual relationship and first numerical properties, 2010), is presented.

Definition 3.1. Let $\acute{C} \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$. A countable family of vectors $\Phi = \{\emptyset_k \in V_{\mathbb{H}}^L : k \in I\}$ is said to be a frame controlled by the operator $\acute{C}$ or the $\acute{C}$– controlled frame if there exist constants $0 < A_{\acute{C}S} \leq B_{\acute{C}S} < \infty$ such that

$$A_{\acute{C}S}\, \|\psi\|^2 \leq \sum_{k\in I}\langle\psi|\emptyset_k\rangle\langle\acute{C}\emptyset_k|\psi\rangle \leq B_{\acute{C}S}\, \|\psi\|^2 \quad \textbf{(3.1)}$$

for all. $\psi \in V_{\mathbb{H}}^L$.

Controlled frame operator can be defined as

$$S_{\acute{C}\psi} = \sum_{k\in I}\langle\psi|\emptyset_k\rangle\acute{C}\emptyset_k. \quad \textbf{(3.2)}$$

Now we have the frame operator

$$S : V_{\mathbb{H}}^L \dashrightarrow V_{\mathbb{H}}^L,\ S\psi = TT^{\dagger}\psi = \sum_{k\in I}\langle\psi|\emptyset_k\rangle\emptyset_k. \quad \textbf{`(3.3)}$$

For $\acute{C}: V_{\mathbb{H}}^L \to V_{\mathbb{H}}^L$, consider

$$\langle\acute{C}S\psi|\psi\rangle = \langle\acute{C}\Big(\sum_{k\in I}\langle\psi|\emptyset_k\rangle\emptyset_k\Big)|\psi\rangle$$
$$= \langle\sum_{k\in I}\langle\psi|\emptyset_k\rangle\acute{C}\emptyset_k|\,\psi\rangle$$
$$= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\langle\acute{C}\emptyset_k|\psi\rangle$$

Now equation 3.1 becomes

$$A_{\acute{C}S}\,\|\psi\|^2 \leq \langle\acute{C}S\psi|\psi\rangle \leq B_{\acute{C}S}\,\|\psi\|^2 \quad \textbf{(3.4)}$$

for all $\psi \in V_{\mathbb{H}}^L$. That is, there exist constants $0 < A_{\acute{C}S} \leq B_{\acute{C}S} < \infty$ such that

$$A_{\acute{C}S}\, I_{V_{\mathbb{H}}^L} \leq \acute{C}S \leq B_{\acute{C}S}\, I_{V_{\mathbb{H}}^L}$$

From Proposition 2.7, $\acute{C}S \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$, and the definition 3.1 is clearly equivalent to $\acute{C}S \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$.

Proposition 3.2. Let $\Phi = \{\emptyset_k \in V_{\mathbb{H}}^L : k \in I\}$ be a $\acute{C}$ – controlled frame in $V_{\mathbb{H}}^L$ for $\acute{C} \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Then Φ is a frame in V_H^L. Moreover $\acute{C}S = S\acute{C}^{\dagger}$ and

$$\sum_{k\in I}\big\langle\psi|\emptyset_k\big\rangle\acute{C}\emptyset_k = \sum_{k\in I}\big\langle\psi|\acute{C}\emptyset_k\big\rangle\emptyset_k,$$

for all $\psi \in V_{\mathbb{H}}^L$.

Proof. Let $\{\emptyset_k\}_{k\in I}$ be a $\acute{C}$ – controlled frame in $V_{\mathbb{H}}^L$ for $\acute{C} \in \mathcal{GL}(V_{\mathbb{H}}^L)$. Then there exist constants $0 < A_{\acute{C}S} \leq B_{\acute{C}S} < \infty$ such that

$$A_{\acute{C}S}\,\|\psi\|^2 \leq \sum_{k\in I}\langle\psi|\emptyset_k\rangle\langle\acute{C}\emptyset_k|\psi\rangle \leq B_{\acute{C}S}\,\|\psi\|^2\ , \quad \textbf{(3.5)}$$

for all $\psi \in V_{\mathbb{H}}^{L}$. It follows that

$$A_{\mathcal{C}S} \langle I_{V_H^L}\psi|\psi\rangle \le \langle S_{\mathcal{C}}\psi|\psi\rangle \le B_{\mathcal{C}S} \langle I_{V_H^L}\psi|\psi\rangle \quad \textbf{(3.6)}$$

for all $\psi \in V_{\mathbb{H}}^{L}$.

Hence $A_{\mathcal{C}S} I_{V_{\mathbb{H}}^L} \le S_{\mathcal{C}} \le B_{\mathcal{C}S} I_{V_{\mathbb{H}}^L}$.

From Proposition 2.7,

$$S_{\mathcal{C}} \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L}).$$

Define $\hat{S} = \mathcal{C}^{-1}S_{\mathcal{C}}$.. Then $\hat{S} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ as $\mathcal{C}^{-1}$, $S_{\mathcal{C}} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$.

Let $\psi \in V_{\mathbb{H}}^{L}$. Then

$$\begin{aligned}\hat{S}\psi = \mathcal{C}^{-1}S_{\mathcal{C}}\psi &= \mathcal{C}^{-1}\left(\sum_{k\in I}\langle\psi|\emptyset_k\rangle\mathcal{C}\emptyset_k\right)\\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\mathcal{C}^{-1}\mathcal{C}\emptyset_k\\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle I_{V_H^L}\emptyset_k\\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\emptyset_k = S\psi\end{aligned}$$

Hence S is everywhere defined and $S \in \mathcal{GL}(V_{\mathbb{H}}^{L})$. Thereby Φ is a frame in $V_{\mathbb{H}}^{L}$.

Since $S_{\mathcal{C}} \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L})$, $S_{\mathcal{C}}$ is positive, therefore $S_{\mathcal{C}}$ is self-adjoint.

For $\psi \in V_{\mathbb{H}}^{L}$,

$$\begin{aligned}\mathcal{C}S\psi &= \mathcal{C}\left(\sum_{k\in I}\langle\psi|\emptyset_k\rangle\emptyset_k\right)\\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\mathcal{C}\emptyset_k\\ &= S_{\mathcal{C}}\psi\end{aligned}$$

Hence $\mathcal{C}S = S_{\mathcal{C}}$.

Now $S_{\mathcal{C}}^{\dagger} = (\mathcal{C}S)^{\dagger} = S^{\dagger}\mathcal{C}^{\dagger} = S\mathcal{C}^{\dagger}$. But $S_{\mathcal{C}}^{\dagger} = S_{\mathcal{C}}$ as $S_{\mathcal{C}}$ is self-adjoint. Therefore $S_{\mathcal{C}} = S_{\mathcal{C}}^{\dagger}$.

It follows that $\mathcal{C}S = S\mathcal{C}^{\dagger}$

Also for $\psi \in V_{\mathbb{H}}^{L}$, $\mathcal{C}S\psi = S\mathcal{C}^{\dagger}\psi$ and $S_{\mathcal{C}}\psi = S\mathcal{C}^{\dagger}\psi$. Hence

$$\begin{aligned}\sum_{k\in I}\langle\psi|\emptyset_k\rangle\mathcal{C}\emptyset_k = S_{\mathcal{C}}\psi = S\mathcal{C}^{\dagger}\psi\\ &= \sum_{k\in I}\langle\mathcal{C}^{\dagger}\psi|\emptyset_k\rangle\emptyset_k\\ &= \sum_{k\in I}\langle\psi|\mathcal{C}\emptyset_k\rangle\emptyset_k\end{aligned}$$

Thereby $\sum_{k\in I}\langle\psi|\emptyset_k\rangle\mathcal{C}\emptyset_k = \sum_{k\in I}\langle\psi|\mathcal{C}\emptyset_k\rangle\emptyset_k$

The above Proposition shows that every controlled frame is a usual frame. But if $\mathcal{C} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ is self-adjoint, we can give a necessary and sufficient condition for a frame to be a controlled frame and vice-versa.

Proposition 3.3. Let $\mathcal{C} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$ be self-adjoint. The family $\{\emptyset_k\}_{k\in I}$ is a frame controlled by $\mathcal{C}$ if and only if it is a frame in $V_{\mathbb{H}}^{L}$ and $\mathcal{C}$ is positive and commutes with the frame operator S.

Proof. Let $\mathcal{C} \in \mathcal{GL}(V_{\mathbb{H}}^{L})$be self-adjoint. Suppose that $\{\emptyset_k\}_{k\in I}$is a frame controlled by $\mathcal{C}$. Then from Proposition 3.2, $\{\emptyset_k\}_{k\in I}$ is a frame in $V_{\mathbb{H}}^{L}$ and $\mathcal{C}S = S\mathcal{C}^{\dagger}$. Since $\mathcal{C}$ is selfadjoint, $\mathcal{C} = \mathcal{C}^{\dagger}$. Hence $\mathcal{C}S = S\mathcal{C}$. Thereby $\mathcal{C}$ commutes with the frame operator S. It follows that $\mathcal{C} = S\mathcal{C}S^{-1} = S_{\mathcal{C}}S^{-1}$ and $\mathcal{C}$ is positive.

On the other hand suppose that $\{\emptyset_k\}_{k\in I}$ is a frame in $V_{\mathbb{H}}^{L}$ with frame operator S and $\mathcal{C}$ is positive and commutes with S. Then $S \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L})$. Therefore $\mathcal{C}S = S\mathcal{C} \in \mathcal{GL}^{+}(V_{\mathbb{H}}^{L})$ and so $S_{\mathcal{C}}$ is positive.

From Proposition 2.7, there exist $A > 0$ and $B < \infty$ such that

$$AI_{V_{\mathbb{H}}^L} \leq S_{C} \leq BI_{V_{\mathbb{H}}^L}. \quad \textbf{(3.7)}$$

For $\psi \in V_{\mathbb{H}}^L$, 3.7 becomes

$$\left\langle AI_{V_{\mathbb{H}}^L}\psi \middle| \psi\right\rangle \leq \langle S_{C}\psi|\psi\rangle \leq \left\langle BI_{V_{\mathbb{H}}^L}\psi \middle| \psi\right\rangle. \quad \textbf{(3.8)}$$

It follows that

$$A\|\psi\|^2 \leq \sum_{k\in I}\langle\psi|\emptyset_k\rangle\langle C\emptyset_k|\psi\rangle \leq B\|\psi\|^2 \quad \textbf{(3.9)}$$

Hence $\{\emptyset_k\}_{k\in I}$ is a frame controlled by C.

4 Weighted frames and frame multipliers in $V_{\mathbb{H}}^L$

In this section we present connection between frame multipliers and weighted frames in a left quaternionic Hilbert space.

Definition 4.1. Let $\{\emptyset_k\}_{k\in I}$ be a sequence of elements in $V_{\mathbb{H}}^L$ and $\{\omega_k\}_{k\in I} \subseteq \mathbb{R}^+$ sequence of positive weights. This pair is called a $\omega-$ frame for $V_{\mathbb{H}}^L$ if there exist constants $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum \omega_k|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2 \quad \textbf{(4.1)}$$

for all $\psi \in V_{\mathbb{H}}^L$.

Definition 4.2. A sequence $\{\zeta_n\}$ is called semi-normalized if there are bounds $b \geq a > 0$ such that $a \leq |\zeta_n| \leq b$.

Definition 4.3. Let, $U_{\mathbb{H}}^L, V_{\mathbb{H}}^L$ be left quaternionic Hilbert spaces, let $\{\psi_k\}_{k\in I} \subseteq U_{\mathbb{H}}^L$ and $\{\emptyset_k\}_{k\in I} \subseteq V_{\mathbb{H}}^L$ be frames. Fix $\boldsymbol{m} = \{m_k\} \in l^\infty(I)$. Define the operator $\mathbf{M}_{\mathbf{m},\{\emptyset_k\},\{\psi_k\}} : U_{\mathbb{H}}^L \to V_{\mathbb{H}}^L$ the *frame multiplier* for the frames $\{\psi_k\}$ and $\{\emptyset_k\}$ as the operator

$$\mathbf{M}_{\mathbf{m},\{\emptyset_k\},\{\psi_k\}}(h) = \sum_{k\in I} \mathrm{m_k}\langle h|\psi_k\rangle\emptyset_k\,;\ h \in U_{\mathbb{H}}^L. \quad \textbf{(4.2)}$$

The sequence $\mathbf{m}$ is called the *symbol* of $\mathbf{M}$. We will denote $\mathbf{M}_{\mathbf{m},\{\emptyset_k\}} = \mathbf{M}_{\mathbf{m},\{\emptyset_k\},\{\emptyset_k\}}$.

Proposition 4.4. Let $C \in \mathcal{GL}\left(V_{\mathbb{H}}^L\right)$ be self-adjoint and diagonal on $\Phi = \{\emptyset_k\}_{k\in I}$ and assume it generates a controlled frame. Then the sequence $\{\omega_k\}$ which verifies the relation $C\emptyset_k = \omega_k\emptyset_k$ is semi normalized and positive. Furthermore $C = M_{\omega,\widetilde{\Phi},\Phi}$, where $\Phi = \{L^{-1}\emptyset_k\}_{k\in I}$ and L is the frame operator for Φ.

Proof. Since $C \in \mathcal{GL}(V_{\mathbb{H}}^L)$ is self-adjoint and $\Phi = \{\emptyset_k\}$ is a frame controlled by the operator C, from Proposition 3.3, C is positive. Thereby $C \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$. Since $C \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$, from Proposition 2.7, there exists $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \left\|C^{\frac{1}{2}}\psi\right\|^2 \leq B\|\psi\|^2 \quad \textbf{(4.3)}$$

for all $\psi \in V_{\mathbb{H}}^L$. Since $\emptyset_k = \omega_k\emptyset_k$, $C^{\frac{1}{2}}\emptyset_k = \sqrt{\omega_k}\emptyset_k$. Now equation 4.3 gives

$$0 < A \leq \omega_k \leq B. \quad \textbf{(4.4)}$$

Hence the sequence $\{\omega_k\}$ is positive and semi-normalized.

Since $C \in \mathcal{GL}(V_{\mathbb{H}}^L)$ is self-adjoint and $\Phi = \{\emptyset_k\}$ is a frame controlled by the operator C, from Proposition 3.3, $\Phi = \{\emptyset_k\}$is a frame in $V_{\mathbb{H}}^L$.

Let $\psi \in V_{\mathbb{H}}^L$ then, by Proposition 2.8, $\psi = \sum_{k\in I}\langle\psi|\emptyset_k\rangle L^{-1}\emptyset_k$, where L is the frame operator for $\Phi = \{\emptyset_k\}$.

Now

$$\begin{aligned} M_{\omega,\widetilde{\Phi},\Phi}\psi &= \sum_{k\in I}\omega_k\langle\psi|\emptyset_k\rangle\widetilde{\emptyset}_k \\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle\omega_k\widetilde{\emptyset}_k \text{ as } \omega_k \text{ is real} \\ &= \sum_{k\in I}\langle\psi|\emptyset_k\rangle C\widetilde{\emptyset}_k \\ &= C\left(\sum_{k\in I}\langle\psi|\emptyset_k\rangle\widetilde{\emptyset}_k\right) \end{aligned}$$

$$= C\left(\sum_{k\in I}\langle\psi|\emptyset_k\rangle L^{-1}\,\widetilde{\emptyset}_k\right) = C\psi$$

Hence $C = M_{\omega,\widetilde{\Phi},\Phi}$.

Lemma 4.5. *Let* $\{\omega_k\}$ *be a semi normalized sequence with bounds a and b. If* $\{\emptyset_k\}$ *is a frame with bounds A and B then* $\{\omega_k\emptyset_k\}$ *is also a frame with bounds* a^2A *and* b^2B.

Proof. Since $\{\omega_k\}$ is semi normalized, there exists $b \geq a > 0$ such that

$$a \leq |\omega_k| \leq b. \tag{4.5}$$

Since $\{\varphi_k\}$ is a frame with bounds A and B,

$$A\|\psi\|^2 \leq \sum_{k\in I}|\psi|\emptyset_k|^2 \leq B\|\psi\|^2 \tag{4.6}$$

for all $\psi \in V_{\mathbb{H}}^L$.

Let $\psi \in V_{\mathbb{H}}^L$ then $|\langle\psi|\omega_k\emptyset_k\rangle|^2 = |\omega_k|^2\,|\langle\psi|\emptyset_k\rangle|^2$. Thereby

$$\sum_{k\in I}|\langle\psi|\omega_k\emptyset_k\rangle|^2 = \sum_{k\in I}|\omega_k|^2|\langle\psi|\emptyset_k\rangle|^2$$
$$\leq b^2\sum_{k\in I}|\langle\psi|\emptyset_k\rangle|^2$$
$$\leq b^2B\|\psi\|^2$$

Similarly one can prove that

$$\sum_{k\in I}|\langle\psi|\omega_k\emptyset_k\rangle|^2 \geq a^2A\|\psi\|^2$$

Hence

$$a^2A\|\psi\|^2 \leq \sum_{k\in I}|\langle\psi|\omega_k\emptyset_k\rangle|^2 \leq b^2B\|\psi\|^2$$

for all $\psi \in V_{\mathbb{H}}^L$.

It follows that $\{\omega_k\varphi_k\}$ is a frame in $V_{\mathbb{H}}^L$ with frame bounds a^2A and b^2B.

Lemma 4.6. *Let* $\Phi = \{\emptyset_k\}$ *be a frame for* $V_{\mathbb{H}}^L$. *Let* $\boldsymbol{m} = \{m_k\}$ *be a positive semi-normalized sequence. Then the multiplier* $M_{\boldsymbol{m},\Phi}$ *is the frame operator of the frame* $\{\sqrt{m_k}\varphi_k\}$ *and therefore it is positive, self-adjoint and invertible. If* $\{m_k\}$ *is negative and semi-normalized then* $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ *is negative, self-adjoint and invertible.*

Proof. We have the frame multiplier for the frame $\Phi = \{\emptyset_k\}$

$$\boldsymbol{M}_{\boldsymbol{m},\Phi}\,\psi = \sum m_k\,\langle\psi|\emptyset_k\rangle\emptyset_k, \tag{4.7}$$

where $\boldsymbol{m} = \{m_k\}$ is the weight sequence.

Since $\{m_k\}$ is semi-normalized sequence *and* $\Phi = \{\emptyset_k\}$ is a frame for, from Lemma 4.5, $\{\sqrt{m_k}\emptyset_k\}$ is a frame for $V_{\mathbb{H}}^L$

Let $\psi \in V_{\mathbb{H}}^L$ then

$$\boldsymbol{M}_{\boldsymbol{m},\Phi}\psi = \sum_k m_k\langle\psi|\emptyset_k\rangle\emptyset_k$$
$$= \sum_k\langle\psi|\sqrt{m_km_k}\emptyset_k\rangle\emptyset_k$$
$$= \sum_k\langle\psi|\sqrt{m_k}\emptyset_k\rangle\sqrt{m_k}\emptyset_k$$
$$= S_{\sqrt{m_k}\emptyset_k}\psi$$

where $S_{\sqrt{m_k}\emptyset_k}$ is the frame operator for the frame $\{\sqrt{m_k}\emptyset_k\}$.

Hence the frame multiplier $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is the frame operator of the frame $\{\sqrt{m_k}\emptyset_k\}$.

Since the frame operator is always positive, self-adjoint and invertible, $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is positive, self-adjoint and invertible.

If $\{m_k\}$ is negative then $m_k < 0$, for all k. So that $m_k = -\sqrt{|m_k|^2}$. Thereby

$$\begin{aligned}\boldsymbol{M}_{\boldsymbol{m},\Phi}\psi &= -\sum_k \sqrt{|m_k|^2}\langle\psi|\emptyset_k\rangle\emptyset_k \\ &= -\sum_k \langle\psi|\sqrt{|m_k|^2}\emptyset_k\rangle\,\emptyset_k \\ &= -\sum_k \langle\psi|\sqrt{|m_k|}\emptyset_k\rangle\sqrt{|m_k|}\emptyset_k \\ &= -S_{\sqrt{|m_k|}\emptyset_k}\psi\end{aligned}$$

Hence $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is negative, self-adjoint and invertible as the frame operator $S_{\sqrt{|m_k|}\emptyset_k}$ is always positive, self-adjoint and invertible.

Theorem 4.7. *Let $\Phi = \{\emptyset_k\}$ be a sequence of elements in $V_{\mathbb{H}}^L$. Let $\{\omega_k\}$ be a sequence of positive,semi-normalized weights. Then the following conditions are equivalent:*

i. *$\{\emptyset_k\}$ is a frame.*
ii. *$\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is a positive and invertible operator*
iii. *There are constants $A > 0$ and $B < \infty$ such that*
$$A\|\psi\|^2 \leq \sum_{k\in I}\omega_k|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2, \forall\, \psi \in V_{\mathbb{H}}^L$$
iv. *$\{\sqrt{\omega_k}\emptyset_k\}$ is a frame.*
v. *$\{\omega_k\emptyset_k\}$ is a frame. i.e., the pair $\{\omega_k\},\{\emptyset_k\}$ forms a weighted frame.*

Proof. For $i. \Rightarrow ii.$, suppose that $\{\emptyset_k\}$ is a frame in $V_{\mathbb{H}}^L$. From Lemma 4.6, the multiplier $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is a frame operator of the frame $\{\sqrt{m_k}\varphi_k\}$ and therefore it is positive and invertible.

For ii.⇔ iii., suppose that $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is a positive and invertible operator. Then $\boldsymbol{M}_{\boldsymbol{m},\Phi} \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$ From Proposition 2.7, there exists $A > 0$ and $B < \infty$ such that $AI_{V_{\mathbb{H}}^L} \leq \boldsymbol{M}_{\boldsymbol{m},\Phi} \leq BI_{V_{\mathbb{H}}^L}$. It follows that for $\psi \in V_{\mathbb{H}}^L$,

$$A\|\psi\|^2 \leq \langle \boldsymbol{M}_{\boldsymbol{m},\Phi}\,\psi|\psi\rangle \leq B\|\psi\|^2$$

If we take $\boldsymbol{m} = \{\omega_k\}$ then

$$A\|\psi\|^2 \leq \langle\sum\omega_k\langle\psi|\emptyset_k\rangle\emptyset_k|\psi\rangle \leq B\|\psi\|^2$$

and

$$A\|\psi\|^2 \leq \sum\omega_k|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2$$

So that there are constants $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum\omega_k|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.8)}$$

i.e. the pair $\{\omega_k\},\{\emptyset_k\}$ forms a $\omega-$ frame.

On the other hand suppose that there exist constants $A > 0$ and $B < \infty$ such that $A\|\psi\|^2 \leq \sum\omega_k|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2$. It follows that $A\|\psi\|^2 \leq \langle\boldsymbol{M}_{\boldsymbol{m},\Phi}\,\psi|\psi\rangle \leq B\|\psi\|^2$, for all $\psi \in V_{\mathbb{H}}^L$. Hence $A \leq I_{V_{\mathbb{H}}^L} \leq \boldsymbol{M}_{\boldsymbol{m},\Phi}\, BI_{V_{\mathbb{H}}^L}$. From Proposition 2.7, $\boldsymbol{M}_{\boldsymbol{m},\boldsymbol{\Phi}} \in \mathcal{GL}^+(V_{\mathbb{H}}^L)$.

Thereby $\boldsymbol{M}_{\boldsymbol{m},\Phi}$ is positive and invertible.

For $iii \Leftrightarrow iv$, suppose that there exist constants $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum\omega_k|\langle f|\emptyset_k\rangle|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.9)}$$

for all $\psi \in V_{\mathbb{H}}^L$. Thereby

$$A\|\psi\|^2 \leq \sum\left|\langle\psi\middle|\sqrt{\omega_k}\emptyset_k\rangle\right|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.10)}$$

for all $\psi \in V_{\mathbb{H}}^L$, as $\{\omega_k\}$ is the sequence of positive weight. Hence $\{\sqrt{\omega_k}\psi_k\}$ is a frame in for all $\psi \in V_{\mathbb{H}}^L$. In similar argument the converse part can be obtained.

For $i \Leftrightarrow iv$, suppose that $\{\emptyset_k\}$ is a frame and $\{\omega_k\}$ is a sequence of positive, seminormalized weights. Then there exist constants $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum|\langle\psi|\emptyset_k\rangle|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.11)}$$

for all $\psi \in V_{\mathbb{H}}^L$ and there exist constants $b \geq a > 0$ such that

$$a \leq |\omega_k| \leq b. \qquad \textbf{(4.12)}$$

Let $\psi \in V_{\mathbb{H}}^{L}$, from equations 4.11 and 4.12

$$\sum \left|\langle \psi | \sqrt{\omega_k} \emptyset_k \rangle\right|^2 = \sum |\omega_k| |\langle \psi | \emptyset_k \rangle|^2$$
$$\leq bB\|\psi\|^2$$

In similar way one can prove that $\sum \left|\langle \psi | \sqrt{\omega_k} \emptyset_k \rangle\right|^2 \geq aA\|\psi\|^2$. Hence

$$aA\|\psi\|^2 \leq \sum \left|\langle \psi | \sqrt{\omega_k} \emptyset_k \rangle\right|^2 \leq bB\|\psi\|^2$$

for all $\psi \in V_{\mathbb{H}}^{L}$. Thereby $\{\sqrt{\omega_k} \emptyset_k\}$ is a frame in $V_{\mathbb{H}}^{L}$with bounds aA and bB.

On the other hand suppose that $\{\sqrt{\omega_k} \emptyset_k\}$ is a frame. Then there exist constants $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum \left|\langle \psi | \sqrt{\omega_k} \emptyset_k \rangle\right|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.13)}$$

for all $\psi \in V_{\mathbb{H}}^{L}$. It is clear that if $\{\mathrm{w}_k\}$ is positive and semi-normalized then $\{\mathrm{w}_k{}^{-1}\}$ is also positive and semi-normalized. Then there are bounds $s \geq r > 0$ such that

$$r \leq |\omega_k{}^{-1}| \leq s \qquad \textbf{(4.14)}$$

Let $\psi \in V_{\mathbb{H}}^{L}$, from equations 4.13 and 4.14

$$\sum |\langle \psi | \emptyset_k \rangle|^2 = \sum \left|\left\langle \psi \middle| \sqrt{\omega_k \mathrm{w}_k^{-1}} \emptyset_k \right\rangle\right|^2$$
$$= \sum |\mathrm{w}_k^{-1}| \left|\langle \psi | \sqrt{\omega_k} \emptyset_k \rangle\right|^2$$
$$\leq sB|\psi\|^2$$

Similarly one can get $\sum |\langle \psi | \emptyset_k \rangle|^2 \geq rA|\psi\|^2$. It follows that

$$rA\|\psi\|^2 \leq \sum |\psi | \emptyset_k|^2 \leq sB\|\psi\|^2 \qquad \textbf{(4.15)}$$

for all $\psi \in V_{\mathbb{H}}^{L}$. Thereby $\{\emptyset_k\}$ is a frame in $V_{\mathbb{H}}^{L}$ with frame bounds *rA* and *sB.*

For $v \Rightarrow i$, suppose that $\{\omega_k \emptyset_k\}$ is a frame. Then there exists $A > 0$ and $B < \infty$ such that

$$A\|\psi\|^2 \leq \sum |\langle \psi | \omega_k \emptyset_k \rangle|^2 \leq B\|\psi\|^2 \qquad \textbf{(4.16)}$$

for all $\psi \in V_{\mathbb{H}}^{L}$. We know that $\{\omega_k^2\}$ is positive and semi-normalized as $\{\omega_k\}$ is positive and semi-normalized. Thereby there exist constants $v \geq u > 0$ such that

$$u \leq \left|\omega_k^2\right| \leq v \qquad \textbf{(4.17)}$$

Let $\psi \in V_{\mathbb{H}}^{L}$, from equations 4.16 and 4.17,

$$A\|\psi\|^2 \leq \sum |\langle \psi | \omega_k \emptyset_k \rangle|^2$$
$$= \sum |\omega_k|^2 |\langle \psi | \emptyset_k \rangle|^2$$
$$\leq v \sum |\langle \psi | \emptyset_k \rangle|^2$$

Hence $.\frac{A}{v}\|\psi\|^2 \leq \sum |\langle \psi | \emptyset_k \rangle|^2$

Similarly

$$B\|\psi\|^2 \geq \sum |\langle \psi | \omega_k \emptyset_k \rangle|^2 = \sum |\omega_k|^2 |\langle \psi | \emptyset_k \rangle|^2$$
$$\geq u \sum |\langle \psi | \emptyset_k \rangle|^2$$

Hence $\sum |\langle \psi | \emptyset_k \rangle|^2 \leq \frac{B}{u}|\psi|^2$. It follows that

$$\frac{A}{v}\|\psi\|^2 \leq \sum |\langle \psi | \emptyset_k \rangle|^2 \leq \frac{B}{u}\|\psi\|^2$$

for all $\psi \in V_{\mathbb{H}}^{L}$. Therefore $\{\emptyset_k\}$ is a frame in $V_{\mathbb{H}}^{L}$.

$i \Rightarrow v$ follows from Lemma 4.5. This completes the proof.

5 Conclusion

We have shown that, in the discrete case, controlled frames, weighted frames and frame multipliers can be defined in a left quaternionic Hilbert space in almost the same way as they have been defined in a complex Hilbert space. The non-commutativity of quaternions does not cause any significant difficulty in the proofs of the results obtained. However, the complex numbers are two dimensional while the quaternions are four dimensional, hence the structure of quaternionic Hilbert spaces are significantly different from their complex counterparts, and therefore in the application point of view the theory developed in this note may provide some advantages or drawbacks. The applications of the discrete case developed here and the corresponding continuous theory in the quaternionic setting are yet to be worked out.

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Research Article

Population genetic structure of *Anopheles subpictus* species B using *COII* and *Cytb* markers

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Abstract:

Sri Lanka was declared malaria free on 2016 by World Health Organization. But re-introduction of malaria by travellers is possible. So far there is no in-depth research on malaria vectors carried out in Northern Sri Lanka due to the civil war prevailed for nearly three decades. *Anopheles subpictus s.l* exists as a species complex and is regarded as an important malaria vector in Sri Lanka. The present study carried out to analyze the population genetic structure of *Anopheles subpictus* species B/ *Anopheles sundaicus s.l.* using mitochondrial genes, *Cytochrome c oxidase subunit II (COII)* and *Cytochrome b (Cytb)* for the first time in Sri Lanka. Adult Anopheline mosquitos were collected from five study sites located in three districts of Northern Sri Lanka using different collection techniques. Collected samples from each locality that were molecularly confirmed as *An. subpictus* species B by AS-PCR assay were used in generating *COII* and *Cytb* sequences. According to the genetic diversity results the haplotype diversities and nucleotide diversities were high for both *COII* and *Cytb* sequences. Although neutrality test results of both Tajima's *D* and Fu's Fs values were not significant for both of the *COII* and *Cytb* ($p>0.05$) indicating that the populations are evolving neutrally. Haplotype networks created in this study showing the possibility of gene flow among the five studied populations. Further studies are recommended to analyze *An. subpictus* species B/ *An. sundaicus s.l.* samples from other locations or districts of Sri Lanka using different molecular markers as well as from other Southeast Asian countries to establish a population genetic structure and verify the unknown members of these taxa.

Keywords: Population genetic structure, *Cytochrome c oxidase subunit II*, *Cytochrome b*, AS-PCR assay, Haplotype

1 Introduction

Vector control is one of the most successful strategy for the suppression of mosquito-borne diseases. However, public health burden due to vector-borne diseases continues to grow as the current control measures fail to

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achieve their goals. Therefore, there is an urgent need to develop improved control strategies that will remain functional even in the face of growing insecticide and drug resistance (Coleman & Alphey, 2004; Camara *et al*., 2018). In light of this type of problems, there is an urge to give serious consideration offered by genetic control strategies aimed at vector populations (Wilke & Marrelli, 2012).

A successful genetic control strategy requires a need to figure out the population genetic structure and the level of gene flow within and between target mosquito populations (Barnes *et al*., 2017). Population genetic studies might be helpful to investigate the genetic basis of species complex, local adaptation for changing environmental conditions and speciation processes. They evidence a considerable importance in vector control measures (Dia *et al*., 2013; Enayati & Hemingway, 2010). Mitochondrial DNA (mtDNA) has been used extensively over the last three decades in population genetic and phylogenetic studies in a wide range of animals from *Drosophila* to humans (Hlaing *et al*., 2009). Since it is difficult to design universal primers for amplifying specific regions in mtDNA due to high variability, only a few mitochondrial genes such as *12S rDNA, 16S rDNA, Cytb, ND1, COI* and *COII* have been employed in phylogenetic studies.

An. subpictus s.l is an important malaria vector in Sri Lanka and exists as a sibling species complex. Molecular based studies have shown the presence of two genetically distinct forms of this species; *An. subpictus* species A and *An. subpictus* species B (Weeraratne *et al*., 2017; Surendran *et al*., 2013). According to Surendran *et al*., (2013) *An. subpictus* species B are in fact members of the Sundaicus complex based on genetic similarity to *An. sundaicus s.l.* *An. subpictus* species B had been involved in transmitting malaria in the coastal areas of the Puttalam district in the west coast of Sri Lanka (Abhayawardana *et al*., 1996; Chandra *et al*., 2010).

Sri Lanka has been declared as malaria-free country by WHO since 2013. However re-emergence of malaria through infected overseas travelers is possible due to a prevalence of potential malaria vectors in the country. Detailed studies on population genetic structure of each malaria vectors, especially Anophiline sibling species is important to implement immediate vector control measures if malaria is reintroduced into the country. There are limited number of studies have been done to determine the population genetic structure of Sri Lankan *An. subpictus* complex (Weeraratne *et al*., 2017; Surendran *et al*., 2013). To date there are no studies to investigate the genetic diversity and population structure of *An. subpictus* species B/ *An. sundaicus s.l* in Northern Sri Lanka. Thus the present study is undertaken to analyze the population genetic structure of *An. subpictus* species B/ *An. sundaicus s.l* using mitochondrial genes, *Cytochrome c oxidase subunit II (COII)* and *Cytochrome b (Cytb)* for the first time in Sri Lanka.

2 Materials and Methods

2.1 Study locations, sample collection and species identification

Adult anopheline mosquitoes were collected monthly from December 2014 to February 2016 using cattle baited hut (CBH), cattle baited net (CBN) and indoor (IC) collection techniques (Surendran *et al*., 2010; Surendran *et al*., 2013) from three locations in the Jaffna District; Kudathanai (9°44'47.53"N, 80°16'19.99"E), Puloly (9° 45' 17''N, 80° 0' 28''E) and Kantharodai (9° 45' 18'' 80° 0' 29''), one location in Kilinochchi District; Palai (9°36'26.83"N, 80°19'44.39"E) and one location in Mullaitivu District; Karachi kudiruppu (9° 16' 2''N, 80° 48' 32''E) (*Figure 1). The collected anopheline mosquitoes were morphologically identified as *An. subpictus s.l.* using available taxonomic keys (Amerasinghe *et al.,* 1990; Surendran *et al*., 2012)

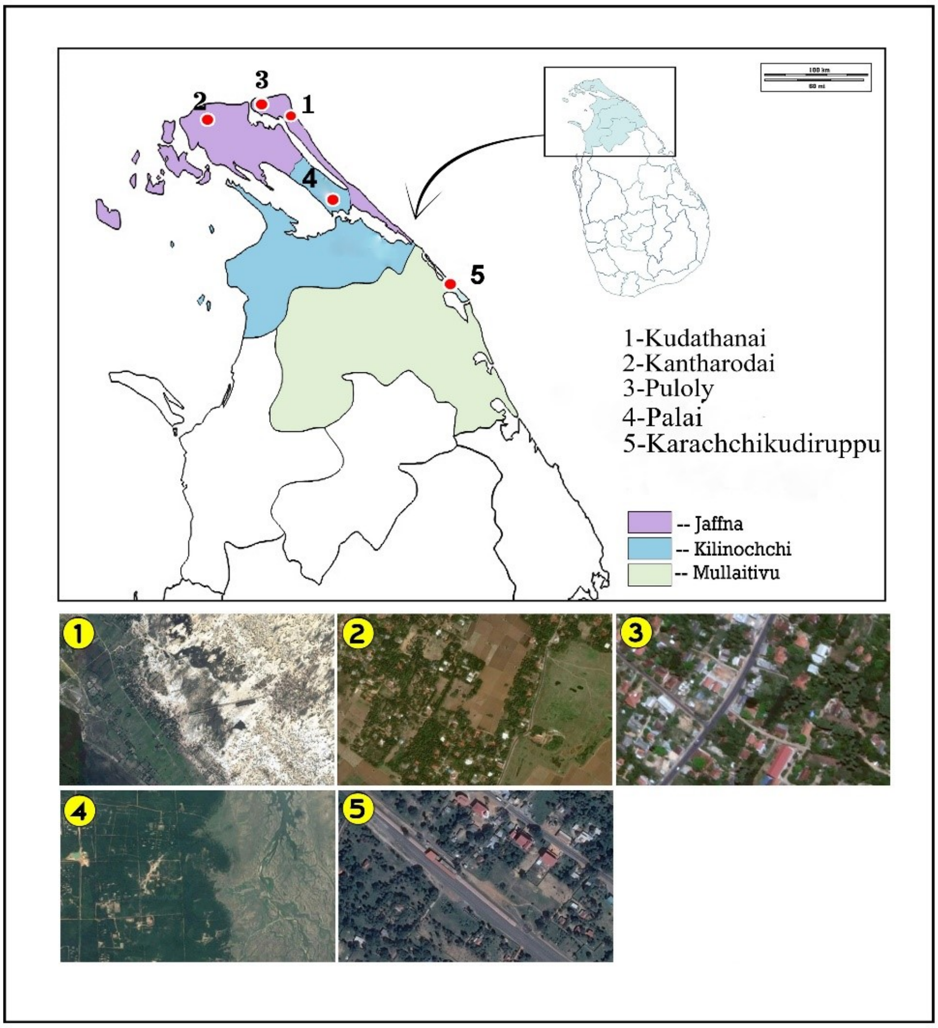

Figure 1: *An. subpictus mosquito sample collection sites*

2.2 Molecular identification of members of Subpictus complex through Allele-specific PCR assay

Genomic DNA was extracted from individual mosquito using Qiagen DNeasy Blood & Tissue Kit (Qiagen, Hilden, Germany) according to the manufacturer's instructions. The extracted DNA was used to screen the mosquitoes for diagnostic allele specific PCR as reported previously (Surendran *et al*., 2013). The diagnostic size of the PCR product for *An. subpictus* species A was ~300 bp while that for *An. subpictus* species B/ *An. sundaicus (s.l.)* was ~400 bp as described in previous studies (Surendran *et al*., 2013). Samples which were identified as *An. subpictus* species B/ *An. sundaicus (s.l.)* were used in the present study.

2.3 Amplification of *COII* and *Cytb* genes and Sequencing

A partial sequence of *Cytb* region of mitochondrial DNA of molecularly identified individual was amplified using forward primer CYTBF[10821] (5'GGACAAATATCATTTTGAGG AGCAACAG-3') and reverse primer CYT BR [11290] (5'ATTACTCCTCCTAGCTTATTAGGAA TTG-3) (Lyman *et al*., 1999). Likewise a partial sequence of the *COII* gene was amplified using forward primer C2J3138 (5'AGAGCCTCTCCTT TAATAGAACA3') and reverse primer C2N368 6 (5'CAATTGGTATAAAACTATGATTTG-3') (Simon *et al.*, 1994). All PCR reactions were performed in the thermal cycler GeneAmp® PCR System 9700. Each PCR (25µl total volume) reaction mixture contained 1 µl of DNA, each primer at 0.5 µM, 2.5 mM $MgCl_2$, 0.2 mM dNTP mix and 1.25 U Taq DNA polymerase in 1x PCR buffer (Bioline, UK).The thermal profile for *Cytb* amplification was 94°C for 4 min, followed by 35 cycles of 94°C for 40 sec, 50°C for 45 sec and 72°C for 45 sec. A final extension temperature of 72°C was set for 10 min. For *COII* amplification samples were initially heated at 94°C for 4 min before 30 cycles of amplification at 95°C for 40 sec, 50°C for 40 sec, and 68°C for 40 sec followed by a final extension at 72°C for 10 min.

The amplified PCR products were visualized by electrophoresis in 1% agarose gels stained with ethidium bromide. PCR products showing positive bands were sent to Macrogen Inc, South Korea, for both the forward and reverse directional sequencing.

2.4 Sequence analysis

Chromatograms were manually edited in FinchTV 1.4.0 (Geospiza, Inc., Seattle, WA, USA; http://www.geospiza.com). Low quality sequences were excluded. All good quality sequences were aligned using Clustal W in BioEdit 7.0.4 software (Tamura *et al*., 2013). Once the alignment was completed, all generated *COII* and *Cytb* sequences were compared along with the publicly available sequence data in GenBank using Basic Logical Alignment Search Tool (BLAST) [www.blast.ncbi.nlm.nih.gov] and the Barcode of Life Data-base (BOLD) interface [www.boldsystems.org] to confirm species identification.

Amino acid sequences were inferred to check for the presence of ambiguous stop codons, genetic information such as number of haplotypes (h), genetic diversity indices ie: Haplotype Diversity Index (Hd) and Nucleotide Diversity Index (Pi) and Neutrality tests (Tajima's *D* and Fu's *Fs*) were determined for both *COII* and *Cytb* sequences using DNA Sequences Polymorphism software (dnaSP) (Version 5.1.10). The pair wise fixation index (F_{ST}) for the identified genetic groups was determined in Arlequin 3.11 and significance was evaluated based on 10000 permutations. Based on the number of nucleotide differences, haplotype networks both *COII* and *Cytb* sequences were constructed using Network software 5.0.0.1 to determine the interrelationship between haplotypes. Corresponding sequences of *COII* and *Cytb* haplotypes were deposited in GenBank database.

3 Results and Discussion

A total of 51 *COII* sequences (~458 bp) and 61 *Cytb* sequences (~383bp) were used for population genetic analysis. The NCBI BLAST confirmed the sequences as belonging to *An. subpictus* species B. Translated amino acid sequences revealed that there is no frame shift or stop codons in all the edited sequences, indicating the mitochondrial origin of DNA. Results of genetic diversity indices and neutrality tests for *An. subpictus* species B/ *An. sundaicus s.l* based on *COII* and *Cytb* sequences are presented in **Table 1**. The overall haplotype diversities (Hd) and nucleotide diversities (Pi) were high in *Cytb* sequences when comparing *COII* sequences.

According to neutrality test results both Tajima's *D* and Fu's Fs values were not significant for both of the *COII* and *Cytb* mitochondrial gene sequences (p>0.05). These results revealed that the populations are evolving neutrally without any purifying selection, population expansion, and selective sweeps (Galtier *et al*., 2000).

Table 1: *Results of genetic diversity indices and neutrality tests for An. subpictus species B/ An. sundaicus s.l. based on COII and Cytb sequences*

Marker	s	h	Hd (±SD)	Pi(±SD)	D	Fs	GenBank accession numbers
COII	19	16	0.890 (± 0.042)	0.00448 (± 0.008)	-1.52811 (p>0.05)	0.34282 (p>0.05)	MK238677-MK238692
Cytb	72	27	0.928 (± 0.002)	0.04252 (0.004)	0.05529 (p>0.05)	1.3132 (p>0.05)	MK238693-MK238719

s- number of polymorphic sites, h- number of haplotypes, Hd- haplotype diversity,
Pi - nucleotide diversity, D- Tajima's *D* and Fs- Fu's Fs.

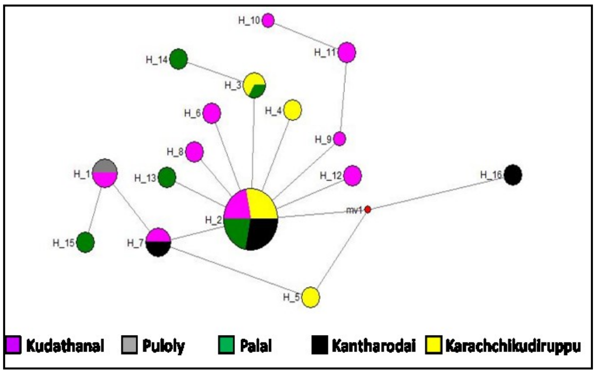

Figure 2: *COII haplotype networks generated using Network 5.0.0.1 for An. subpictus species B/ An. sundaicus s.l collected from five geographical locations in northern Sri Lanka*

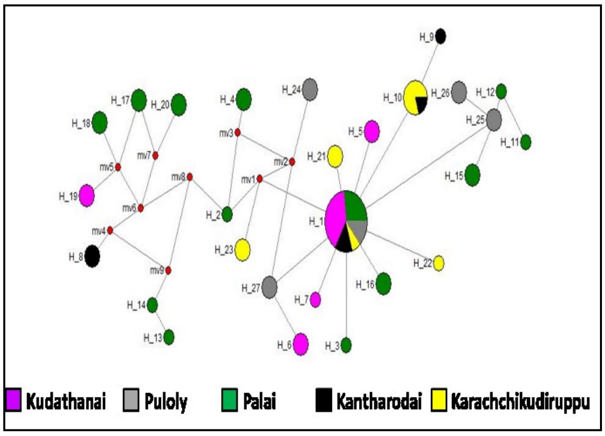

Figure 3: *Cytb haplotype networks generated using Network 5.0.0.1 for An. subpictus species B/ An. sundaicus s.l collected from five geographical locations in northern Sri Lanka*

From 51 *COII* sequences sixteen different haplotypes (H1-H16) were identified (**Table 1**). The haplotype network for *COII* sequences is indicated in ***Figure 2***. There were four shared haplotypes and H2 was the dominant haplotype (35.29% of the total number of haplotypes) shared among populations collected from all localities except Puloly. A total 27 different haplotypes (H1-H27) were identified among the 61 *Cytb* sequences **Table 1** and the haplotype network for *Cytb* sequences is indicated in ***Figure 3***. Among the two shared haplotypes H1 (24.59% of the total number of haplotypes) was the predominant haplotype representing the samples collected from all five localities.

The pairwise comparison of population differentiation based on *COII* and *Cytb* sequences are presented in Error! Reference source not found. and Error! Reference source not found. respectively. The pairwise differences (F_{ST} values) obtained for both *COII* and *Cytb* sequences were not significant (P>0.05) showing that there is no genetic differentiation among the *An. subpictus* species B/ *An. sundaicus s.l* populations from the five sampling localities selected during the current study.

Table 2: *Population pairwise FST values obtained for An. subpictus species B/ An. sundaics s.l samples from the five different geographical locations based on COII sequences*

Location	Palai	Kudathanai	Kantharoda i	Karachchi kudiruppu	Puloly
Palai	0.00000				
Kudathanai	0.07079	0.00000			
Kantharodai	0.08818	0.07240	0.00000		
Karachchi kudiruppu	0.03568	0.06815	0.02439	0.00000	
Puloly	0.54483	0.59626	0.65812	0.70370	0.00000

Table 3: *Population pairwise FST values obtained for An. subpictus species B/ An. sundaics s.l samples from the five different geographical locations based on Cytb sequences.*

Location	Palai	Kudathanai	Kantharodai	Karachchi kudiruppu	Puloly
Palai	0.00000				
Kudathanai	0.29623	0.00000			
Kantharodai	0.00442	0.17396	0.00000		
Karachchi kudiruppu	0.31079	0.12124	0.16817	0.00000	
Puloly	0.30156	0.07530	0.19608	0.18151	0.00000

An. sundaicus s.l is a major malaria vector in the Oriental region, ranging from Southeast Asia, including Myanmar, Thailand, Malaysia, Indonesia, China, and Indochina (Manguin *et al*., 2008; Surendran *et al*., 2010). *An. sundaicus s.l* was also an important vector of malaria along the East coast of India, found abundantly and widely only in the Andaman and Nicobar islands (Nanda *et al*., 2004; Surendran *et al*., 2010). *An. sundaicus s.l* has never previously been identified in Sri Lanka and this taxon is reported to have at least four cytological forms (A-D) and four molecular forms namely *An. sundaicus s.s*., *An. epiroticus*, *An. sundaicus* cytotype D and *An. sundaicus* E (Surendran *et al*., 2010; Bora *et al*., 2009; WHO, 2007). Phylogenetic analysis based on ITS2 and D3 markers conducted by Surendran *et al*., (2010) revealed that samples identified as *An. subpictus* B in the East coast of Sri Lanka and *An. subpictus s.l* identified elsewhere in South-East Asia are members of the Sundaicus complex based on genetic similarity. Many genes of mtDNA has been used as the genetic tools to study the genetic variation and population structure of the Anopheline mosquitoes successfully, such as *cytochrome subunit I* (*COI*), *cytochrome subunit II* (*COII*), *cytochrome B* (*Cytb*) (Bora *et al*., 2009; Weeraratne *et al*., 2017; Feng *et al*., 2017).

This is the first study carried out to analyze the population genetic structure of *An. subpictus* species B/ *An. sundaicus s.l.* from five different geographical areas of northern Sri Lanka based on two mtDNA markers namely *COII* and *Cytb*. According to the genetic diversity results the haplotype diversities and nucleotide diversities were high for both *COII* and *Cytb* sequences in turn suggests that *COII* and *Cytb* genes of mitochondrial DNA can be considered a suitable molecular marker for calculating genetic variation. The high level of genetic diversity points out that due to broad tolerance to environmental and habitat pressure species could maintain a relatively large effective population size. May be due to variations in the evolutionary rates of different molecular markers, the genetic divergence showed different degrees; Pi values as 0.00448 of *COII* gene while as 0.04252 of *Cytb* gene. Haplotype networks created at this study highlight haplotype sharing between the five populations showing the possibility of gene flow among the five studied populations. Present study also emphasis the necessity of population genetic structure study of *An. subpictus* species B from other parts of the Sri Lanka to asses the genetic relationship among geographically isolated populations.

Further no studies have been done so far to analyze the population genetic structure of the *An. subpictus* complex in Southeast Asian countries and rest of the world except Sri Lanka and India. Therefore, it is important to analyze the population genetic parameters of the members of *An. subpictus* complex from other Southeast Asian countries using different molecular markers to determine the taxonomic status of the *An. subpictus* complex.

4 Conclusion

Present study showed that there is a possibility of gene flow among the five geographically different populations of An. subpictus species B/ *An. sundaicus s.l.* It is important to analyze *An. subpictus* species B/ *An. sundaicus* samples from other locations/ districts of Sri Lanka using different molecular markers as well as from other Southeast Asian countries to establish a population genetic structure and verify the unknown members of this taxon

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Research Article

Abstracts published in the *Proceedings of the Jaffna Science Association* (1992-2017): A bibliometric analysis

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Abstract

Bibliometrics offers a powerful set of methods and measures for studying the structure and process of scholarly communication. By employing bibliometric approach, this paper examines the characteristics of the abstracts published in the *Proceedings of the Jaffna Science Association (JSA),* covering a 25-year period. Bibliographical details of 893 abstracts available in 24 volumes of the JSA proceedings were collected, tabulated, analyzed and reported. The attributes such as, distribution of abstracts, authorship pattern, institutional contribution, medium of language, subject coverage, and format of the abstracts were taken into consideration for the bibliometric analysis. Results showed that there is variation in the number of abstracts published per year, ranging from 17 to 74. Besides, 72.79% of the published abstracts belonged to pure and applied sciences. Collaborative authorship is a prominent feature observed in 81.52% of the abstracts, which is the current trend in scholarly communication. Among the contributing institutions, University of Jaffna is at the forefront. Analysis of subject coverage revealed that 62.71% of the abstracts related to the disciplines namely agriculture, biology and biochemistry. In view of enhancing the accessibility, majority of the authors selected English as a channel for their scholarly communication. Furthermore, it was found that 78.72% of the abstracts comply with the guidelines for the abstract layout, prescribed by the JSA. Regarding the length of abstracts, 44.46% fall within the acceptable range of 200-299 words. This bibliometric study of scholarly communication provided a comprehensive view of the dynamical growth of knowledge of the region.

Keywords: Bibliometrics; Scientific production; Bibliometry; Scholarly publications; Bibliometric analysis; Proceedings of the Jaffna Science Association.

1 Introduction

Jaffna Science Association(JSA) is a registered voluntary organization functioning largely in the Northern region of Sri Lanka since 1991, with the objectives of promoting scientific research, disseminating and applying research findings relevant to the region, and imparting advanced scientific and technological information to the community. In view of accomplishing the objective of promoting research, JSA conducts annual research sessions and publishes the abstracts of the research papers (presented at the research sessions) in the *Proceedings of the Jaffna Science Association(JSA).* It seems appropriate to explore selected bibliometric dimensions of the abstracts published in the *Proceedings of the JSA*, at this significant milestone of the JSA.

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This bibliometric snapshot covers the period 1992-2017, and intended to examine the characteristics of the abstracts published in the *Proceedings of the JSA*, with the assistance of bibliometric method.

Bibliometrics is a well-recognized method in the sociology of science, as it employs quantitative methods and structural approaches (Borgman and Furner, 2002). It is based on statistical data regarding publications, citations and other related indicators of scholarly communications (Chuang *et al.,* 2012). Numerous bibliometric studies of scholarly publications related to various disciplines have been reported (Naseer and Mahmood, 2009; Sam, 2008; Tiew *et al*., 2002; Young, 2006, Chuang *et al*., 2012; Thanuskodi, 2011; Ramos *et al*., 2013).

In recent years, quality of research of an individual or an institution is measured by using these studies and growing interest is observed among researchers in developing new scientific indicators capable of facilitating the bibliometric analysis of research publications (Thanuskodi, 2011).

This study was undertaken with the aim of examining the bibliometric characteristics of the abstracts published in the *Proceedings of the JSA* during 1992-2017 with the following specific objectives: to determine the year-wise distribution of abstract; to analyze the authorship pattern and degree of collaboration; to explore the institutional contribution; to study the subject coverage of published abstracts; to examine the medium of language of abstracts and to assess the layout and length of abstracts published.

2 Methodology

Since 1992, abstracts of the research papers presented at the annual research sessions of the JSA are published in the *Proceedings of the JSA* annually, except on a few occasions. Annual sessions of the JSA was not held in 1996 and 2007, due to exodus from Jaffna in October 1995 and ethnic crisis in the region respectively. The *Proceedings of the JSA* were not published in these years. Hence, 893 abstracts available in the *Proceedings of the JSA* (from volume 1 in 1992 to volume 24 in 2017), were considered for this study.

Methodology applied in the present study is bibliometric analysis, which is used to study the bibliographical features of research publications, in order to get an overview of the research output pertaining to a particular topic. It is a quantitative study of various aspects of literature on a topic (Thanuskodi, 2011). Generally, it is used to identify the pattern of publication and authorship, performance of contributing countries, institutions and authors, characteristics of document types, impact factor, number of cited documents, etc. to gain insight into the dynamical growth of knowledge in the areas under consideration (Chuang *et al*., 2012; Thanuskodi, 2011; Tiew, 2006).

In this study, bibliographical details, namely title, author/s, affiliation of author/s, publication year, language, subject, layout and word count related to 893 abstracts available from volume 1 (1992) to volume 24 (2017) of the *Proceedings of the JSA* were collected. Then, the collected data were tabulated and analyzed for reporting observations.

3 Results and Discussion

A total of 893 abstracts published in the *Proceedings of the JSA* during 1992-2017 were analyzed by using bibliometric approach and results are summarized below.

3.1 Year-wise distribution of abstracts

Generally, the abstracts of all research papers presented at the Annual sessions of the JSA are published in the *Proceedings of the* JSA. The number of abstracts published per year during the period under study is illustrated in ***Figure 1***, which ranges from 17 to 74.

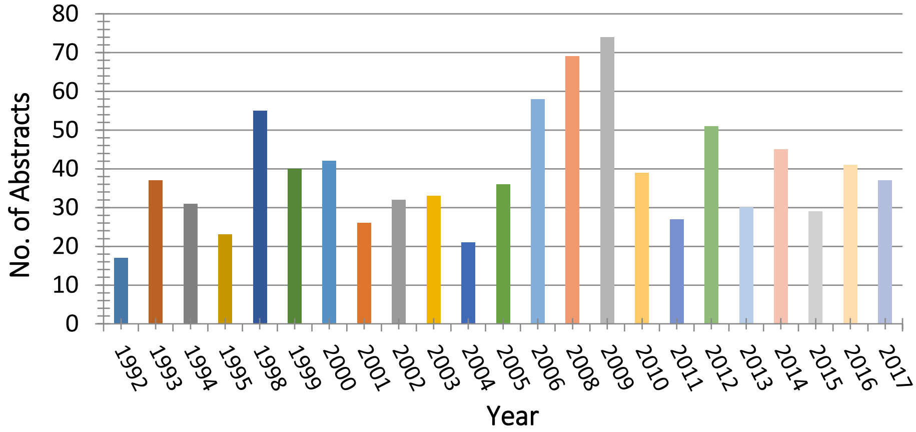

Figure 1: *Year-wise distribution of abstracts published in the Proceedings of the JSA (1992-2017)*

The year-wise distribution revealed that the maximum numbers of abstracts were published in the year 2009 (n=74, 8.29%) and minimum in the year 1992 (n=17, 1.90%). However, progressive increase in the number of abstracts published with time is insignificant.

It is interesting to note that large number of research papers presented at the Annual sessions of the JSA and thereafter published as abstracts in the *Proceedings of the JSA* during the period 2006-2009. This was marked as the end of civil war between the Government of Sri Lanka and LTTE (Liberation Tigers of Tamil Eelam) in the history of Sri Lanka and Northern province was severely affected region in this civil war. In 1998 (after the massive exodus in 1995) too, 55 papers were presented at the Annual sessions of the JSA. These incidences disclose the research potential of the region, in spite of lack of infrastructure facilities, travel restrictions and limitations in food, medicine, etc. However, due to the opportunities (nationally and internationally) available for publishing researches conducted in the region, a decrease in the number of papers presented at the JSA Annual sessions was observed since 2012.

Further, year-wise distribution of papers presented and abstracts published in the *Proceedings of the JSA* under different sections of the JSA, namely Section A (Pure sciences), Section B (Applied sciences), Section C (Medical sciences) and Section D (Social sciences) is shown in ***Figure 2.***

Among the four sections in the JSA, highest number of abstracts (n=424, 47.48%) was submitted in Section B during the entire study period. This was followed by Section A (n=226, 25.31%), Section D (n=124, 13.88%) and Section C (n=119, 13.33%). Moreover, initially more papers were presented in Section A and Section C, during the period 1992 to 1994. Subsequently, Section B dominated in research publication over a period of 16 years (1995-2010). This observation gives an impression that research trend in the region during the period 1995 to 2010, is more oriented towards applied science. Thereafter from 2011 to 2014, the number of research papers related to pure and medical sciences is increasing, while papers in applied sciences is decreasing. Even though very few papers were presented in relation to social sciences initially, there is a steady increase observed since 2012. It reveals that JSA has attained one of its objectives '*promoting scientific research*' in all disciplines through 25 years of its long journey under several hardships.

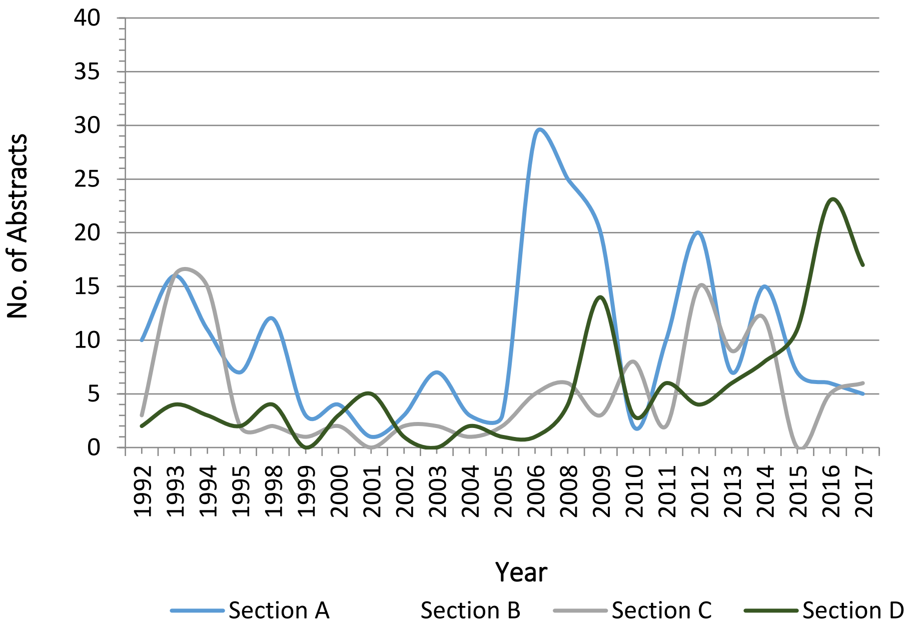

Figure 2: *Year-wise distribution of abstracts published in the Proceedings of the JSA in relation to Sections A, B, C, and D of the JSA (1992-2017)*

3.2 Authorship pattern and Degree of collaboration

Bibliometric studies related to authorship pattern are conducted to analyze the authorship characteristics of a group of authors in a geographical area (Basu and Kumar, 2000; Cheng *et al*., 2013) or related to specific discipline (Maheswarappa and Mathias, 1987; Karisiddappa *et al*., 1990) or type of publication (Tiew, 2006; Haiqi, 1996). Besides, collaboration in scholarly communication is a significant factor, and bibliometric studies revealed that the amount of collaboration between scholars is growing and degree of collaboration varies among disciplines (Arunachalam, 2000; Bordons & Gomez, 2000). Reasons for the growth in collaboration are stated as increasing specialization within disciplines, to meet the expenses of research, and to manage larger projects (Bordons & Gomez, 2000).

The degree of collaboration is defined as the '*ratio of the number of collaborative research papers to the total number of research papers published during a period of time*', which is calculated according to the following formula (Subramanyam, 1983).

$$C = N_m / (N_m+N_s)$$

Where C - Degree of collaboration

N_m - Number of multi-authored research papers published in a year

N_s - Number of single-authored research papers published in a year

The present study showed that significant amount of research was carried out under collaborative authorship (n=728, 81.52%) compared to single authorship (n=165, 18.48%). ***Figure 3*** illustrates the authorship pattern of the abstracts published during the period of study.

This trend of working in collaboration is in conformity with the results of previous studies (Navaneethakrishnan and Sivakumar, 2015; Navaneethakrishnan, 2014). However, it is observed that in certain disciplines (e.g., humanities, library and information science) single authorship is in lead (Tiew, 2006; Naseer and Mahmood, 2009).

Further analysis in collaborative authorship revealed that maximum number of abstracts were contributed by two authors (n=334, 37.4%). This is followed by three authors with 256 abstracts, four authors with 82 abstracts, and five or more authors with 56 abstracts. Further, it was observed that the *Average degree of collaboration* is 0.8 during the study span (***Table 1***).

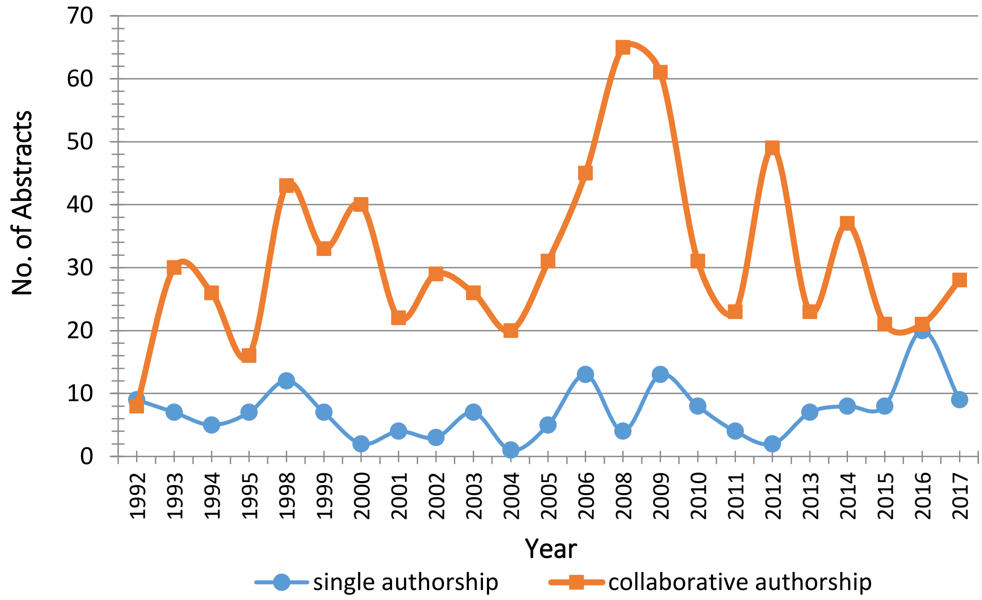

Figure 3: *Year-wise authorship pattern of abstracts published in the Proceedings of the JSA (1992-2017)*

Table 1: *Authorship pattern and degree of collaboration in published abstracts*

Year	Number of Authors						Multi-authored	Degree of Collaboration
	Single	Two	Three	Four	>Five	Total		
1992	9	6	2	0	0	17	8	0.47
1993	7	13	8	5	4	37	30	0.81
1994	5	14	6	5	1	31	26	0.84
1995	7	4	12	0	0	23	16	0.70
1998	12	10	29	4	0	55	43	0.78
1999	7	20	11	2	0	40	33	0.83
2000	2	16	22	2	0	42	40	0.95
2001	4	13	9	0	0	26	22	0.85
2002	3	13	6	10	0	32	29	0.91
2003	7	14	9	3	0	33	26	0.79
2004	1	11	8	0	1	21	20	0.95
2005	5	20	9	0	2	36	31	0.86
2006	13	28	11	3	3	58	45	0.78
2008	4	20	20	12	13	69	65	0.94
2009	13	32	22	7	0	74	61	0.82
2010	8	14	9	0	8	39	31	0.79
2011	4	15	7	1	0	27	23	0.85
2012	2	18	12	9	10	51	49	0.96
2013	7	7	11	3	2	30	23	0.77
2014	8	15	10	5	7	45	37	0.82
2015	8	8	9	3	1	29	21	0.72
2016	20	9	6	4	2	41	21	0.51
2017	9	14	8	4	2	37	28	0.76
Total	**165**	**334**	**256**	**82**	**56**	**893**	**728**	**18.46**

Subsequently, individual authors were ranked in terms of their productivity. ***Table 2*** ranks the six most productive authors based on the number of abstracts published in the *Proceedings of the JSA* during the period 1992 to 2017.

Table 2: *Six most productive authors ranked according to total number of abstracts published in the Proceedings of the JSA (1992-2017)*

Author	Number of Abstracts	Percentage (%)	Rank
Arasaratnam, V.	169	18.92	1
Balakumar, S.	82	9.18	3
Balasubramaniam, K.	78	8.73	2
Senthuran, A.	38	4.26	4
Mikunthan, G.	28	3.14	5
Vasantharuba, S.	28	3.14	5
Total	**423**	**47.37**	

Among the productive authors, V. Arasaratnam (Senior Professor in Biochemistry attached to University of Jaffna) is leading with 169 published abstracts. Followed by S. Balakumar (Senior Lecturer, Department of Biochemistry, UoJ), K. Balasubramaniam (Retired Professor, Department of Biochemistry, UoJ), and A. Senthuran (Former Senior Lecturer, Department of Biochemistry, UoJ) with the contribution of 82, 78 and 38 abstracts respectively. Fifth position is shared by to authors, namely G. Mikunthan (Professor, Department of Agricultural biology, UoJ) and S. Vasantharuba (Senior Lecturer, Department of Agricultural chemistry, UoJ), each with 28 abstracts.

Further, the study demonstrated that 47.37% (n=423) of the abstracts published in the *Proceedings of the JSA* during the period 1992 to 2017 have the contribution of these authors. A distinct feature observed is that all of these most productive authors are attached to the University of Jaffna. In addition, they all are members in Section B of the JSA. This may be one of the reasons for the observation of high number of abstracts in the Section B (JSA).

Moreover, majority of the abstracts appeared in the *Proceedings of the JSA* authored by above mentioned most productive authors were published under collaborative authorship (n=420, 99.3%). Hence, year-wise distribution of abstracts co-authored by most productive authors ranked in positions 1-4 (attached to same department) analyzed and illustrated in ***Figure 4.***

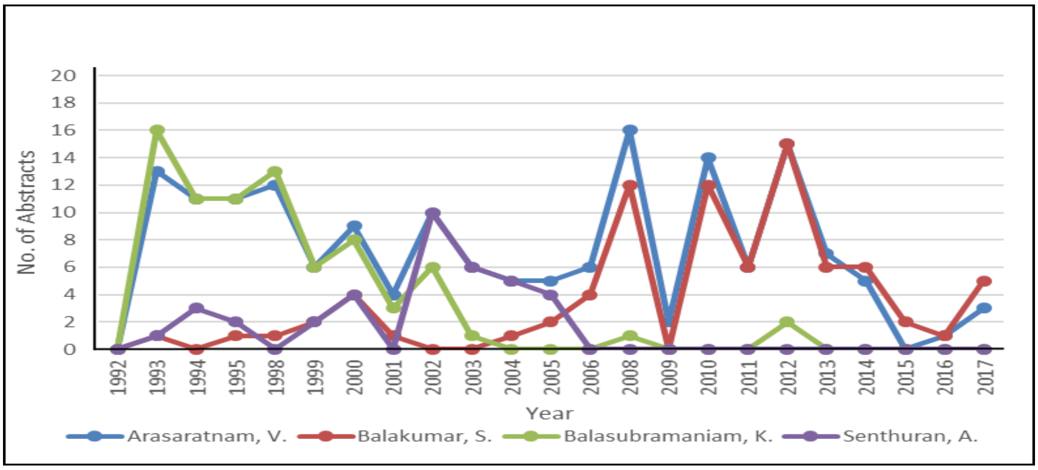

***Figure 4*:** *Year-wise distribution of abstracts co-authored by most productive authors*

Among these most productive authors from the Department of Biochemistry, University of Jaffna, Vasanthy Arasaratnam has contributed in the capacity of co-author throughout the study period (1992-2017) compared to her colleagues.

3.3 Contribution of Institutions

The performance of individual scholars may be aggregated at the level of groups of various sizes, such as research groups, departments, universities, and corporations. By using the author/s affiliation, institutions may be ranked. This would help the university administrators and corporate managers to compare their peers and competitors. Further, government and private funding sources can monitor the return on their investment, and policy makers can identify relative strengths and weaknesses in strategically important sectors (Borgman and Furner, 2002).

The ***Table 3*** envisages the institution-wise contribution where author/s of the abstracts affiliated. These institutions have been grouped into six distinct categories, and ranked based on the number of abstracts contributed by the personnel attached to the relevant category of institutions ***(Table 3).***

Table 3: *Institutional contribution in relation to abstracts published in the Proceedings of the JSA (1992-2017)*

Institutional category	Number of Abstracts	Percentage (%)	Rank
Universities	888	99.4	1
Other Higher Educational Institutions	06	0.7	6
Research institutions	65	7.3	2
Government Departments	19	2.1	4
Non-Governmental Organizations	22	2.5	3
Others	09	1.0	5

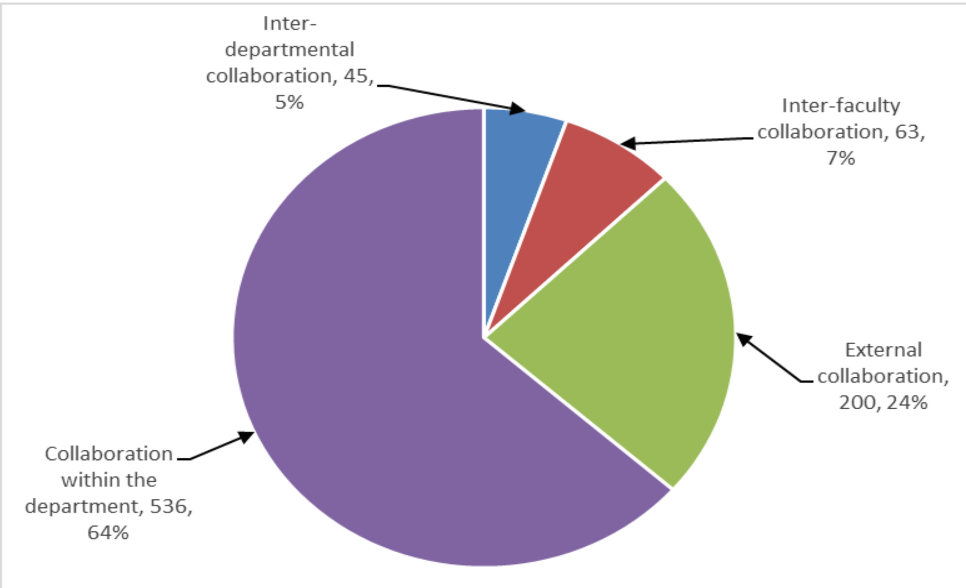

Figure 5: *Different types of collaborations of the University of Jaffna in relation to abstracts published in the Proceedings of the JSA (1992-2017)*

Results showed that highest number of abstracts (n=888, 99.4%) were contributed by the universities in Sri Lanka and foreign countries. Among them, academics attached to University of Jaffna contributed in 94.5% (n=844) of the abstracts, independently or jointly with other institutions. This is followed by contribution from research institutions with 65 abstracts (7.3%), non-governmental organizations with 22 abstracts (2.5%), and government departments with 19 abstracts (2.1%). There were also few contributions from higher educational institutions except universities, namely College of Education, Teachers training colleges, and Technical colleges during the study period. This implies the significant contribution of the University of Jaffna in the research output of the region, between 1992 and 2017. Furthermore, with regard to University of Jaffna, inter-departmental, inter-faculty and external collaborations with other organizations are highlighted in ***Figure 5.***

The ***Figure 5*** indicates that only 36% (n=308) of the research abstracts contributed by the University of Jaffna were produced in collaboration with other departments (within the same faculty) or faculties (departments attached to different faculties of the University of Jaffna) or external sources (such as other universities, research institutions, etc.) during the study period.

3.4 Subject coverage

The abstracts published in the *Proceedings of the JSA* were multidisciplinary in nature. However, compared to pure and applied sciences disciplines number of abstracts related to health and social sciences are limited (***Figure 6)***.

The analysis of subject coverage revealed that 75.59% (n=675) of the total abstracts belonged to pure and applied sciences. Among these, agriculture (n=252, 28.22%), biology (n=155, 17.36%) and biochemistry (n=153, 17.13%) are the key disciplines, where more research has been carried out. Further, 9.07% (n=81) of the abstracts were contributed in health sciences, which includes allied health sciences, western and Siddha medicine. Besides, abstracts in relation to humanities (includes philosophy, linguistics, literature and fine arts), social sciences (includes education, economics, sociology, political science and law) and management are also present. Relatively very few papers on library and information science, sports science, translation studies, media studies and home economics are also available.

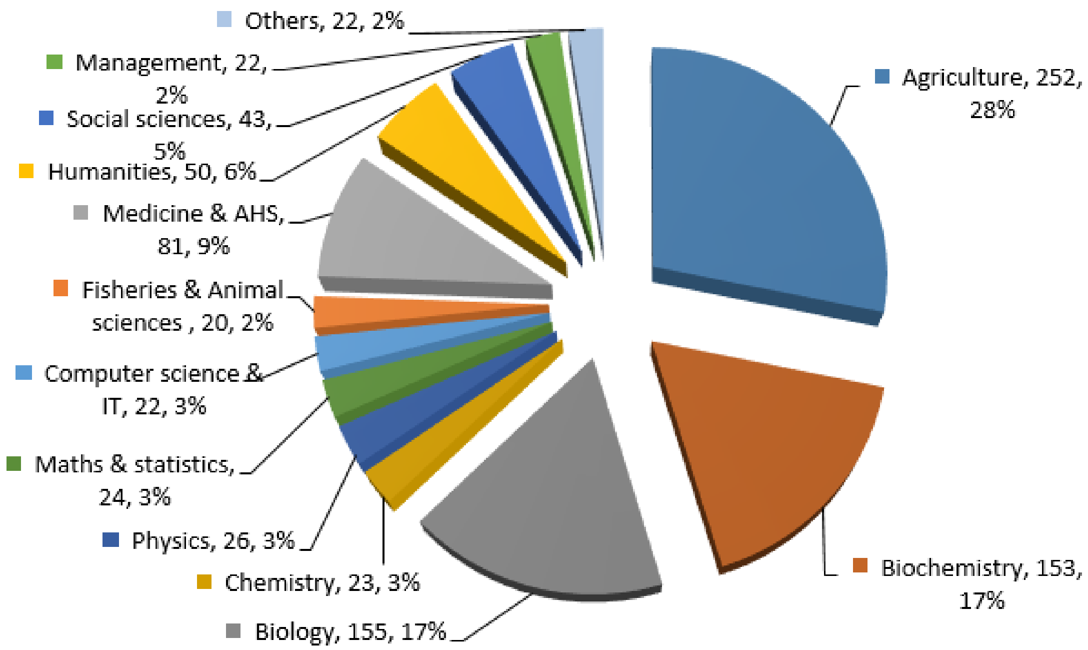

Figure 6 : *Subject coverage of the abstracts published in the Proceedings of the JSA(1992-2017)*

The subject analysis further demonstrated that most of the studies focused and highlighted the issues related to the region. It is noteworthy that these studies were conducted with limited facilities (lack of electricity, inadequate infrastructure facilities, restrictions to transport chemicals and equipments) available in the region during the period of civil war (1980-2009) .

3.5 Language

Regarding the medium of communication, abstracts were written both in English and Tamil languages. Even though English is primarily used as a medium in scholarly communication, research papers in Tamil also entertained by the JSA as it is a regional language and in view of promoting research related to humanities and social sciences in the region. Since, the mother tongue (Tamil or Sinhala) are used as a medium of instruction in most of the humanities and social science disciplines at higher educational level in Sri Lanka, opportunities are limited for them to publish their research findings written in regional languages.

The present study revealed that 95.63% (n=854) of the abstracts were written in English, while 4.37% (n=39) in Tamil language during the study span. This trend observed may be due to the reason that most of the researchers selected English as a channel for scholarly communication in order to enhance the visibility for their publication. Furthermore, year-wise distribution of abstracts written in English and Tamil languages is illustrated in ***Figure 7***.

The results disclose that there was not a single abstract published in Tamil (regional language) during the period 1992 to 2012. However, during the last five years, the number of abstracts published in Tamil language increased from 2 to 15. Moreover, it is worth mentioning that majority of these abstracts were presented in Section D of the JSA, which comprises the disciplines humanities, management and social sciences.

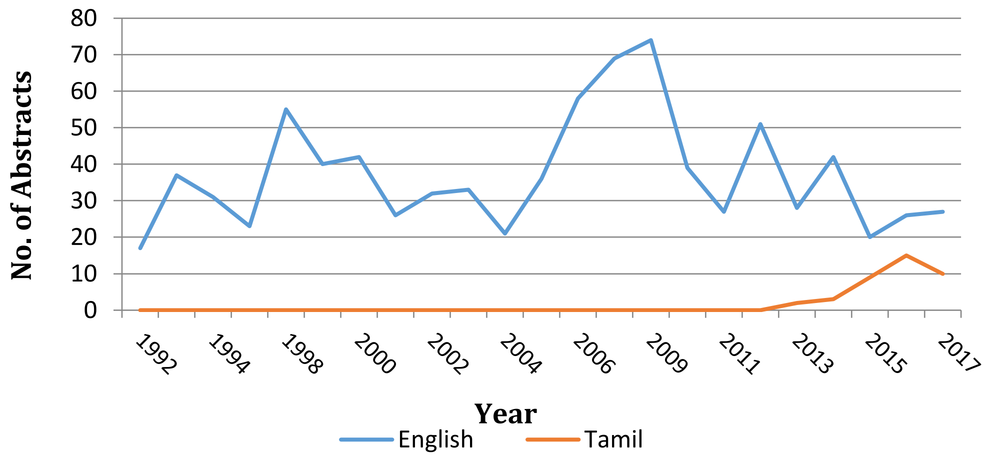

Figure 7: *Year-wise distribution of abstracts published by language in the Proceedings of the JSA (1992-2017)*

3.6 Layout and length of abstracts

According to the guidelines of the JSA, the layout of the abstract should comprise background and objectives, research design, results, discussion and conclusion. The analysis revealed that only 78.72% (n=703) of the abstracts comply with this requirement, while the remaining abstracts are deficient in providing background information or research design or conclusion of the research.

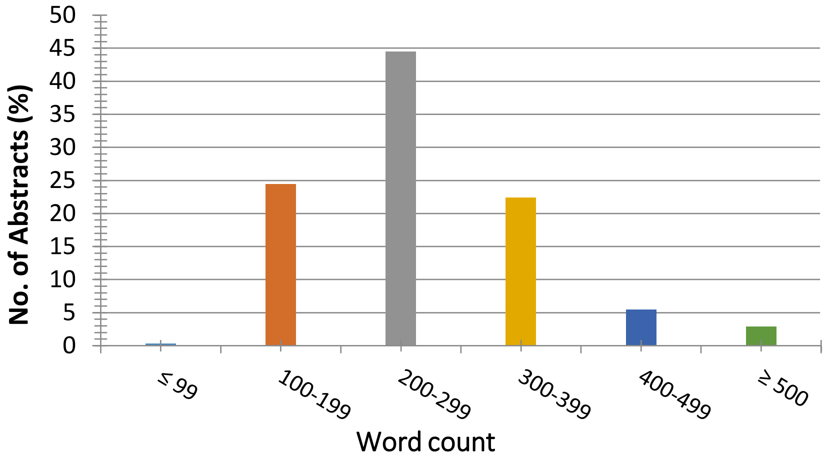

Figure 8: *Length of the abstracts published in the Proceedings of the JSA (1992-2017)*

Besides, length of the abstracts was also measured by using word count. ***Figure 8*** illustrates the number of abstracts distributed within different word count ranges.

The results showed that majority of the abstracts (n=397, 44.46%) consist of words ranging from 200 to 299, which is acceptable for the word limit (250 words) specified in the guidelines of the JSA. Among the remaining, 218 and 200 abstracts found within the word limit 100-199 and 300-399, respectively. Whereas, it was observed that 8.73% (n=78) of the abstracts published in the *Proceedings of the JSA* during the period 1992-2017, fall within the word count range below 99 or above 400, which is unsatisfactory. It is recommended that Editorial team should take necessary steps (during editorial process) to maintain the word count of the abstracts within acceptable limits.

4 Conclusion

A total of 893 abstracts were published in the *Proceedings of the JSA* from 1992 to 2017. Besides, the total number of abstracts published per year ranges from 17 to 74, and there is no steady increase observed with time during the study span. Even though, Section B (Applied sciences) dominated in research publication during the whole study period, there is a steady increase observed in relation to abstracts presented by the Section D (Social sciences) since 2012.

Collaborative authorship pattern found in majority of the abstracts showed the current trend in scholarly communication, due to inter-disciplinary research, sharing of expertise, etc. As a leading higher educational institution in the region, University of Jaffna's contribution (n=844, 94.5%) to research is worth mentioning. Furthermore, agriculture, biology and biochemistry are the leading disciplines where 62.71% of the abstracts published, during the entire study period.

Although English is not a native language of this region, authors prefer to use it in scholarly communication to enhance the visibility of their publications. However, number of abstracts published in Tamil language is increasing since 2012 and majority of these abstracts were presented in Section D of the JSA, which comprises the disciplines humanities, management and social sciences.

Regarding the layout of the abstract, 78.72% of the abstracts comply with the guidelines prescribed by the JSA, while 44.46% of the abstracts satisfy the word limit requirement. This shows that more concern is required with regard to word limit.

This study presented a comprehensive analysis on bibliometric characteristics of the abstracts published in the *Proceedings of the JSA* (1992-2017). Findings of this bibliometric study indicate the distribution and growth of scientific production of the region, which would help the Jaffna Science Association to envisage and move towards future goals.

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